Homework ① solutions

1. (a) It's enough to show that $\{\underset{\sim}{g}_1, \underset{\sim}{g}_2, \underset{\sim}{g}_3\}$ is linearly independent:

$$c^i \underset{\sim}{g}_i = \underset{\sim}{0}$$

$$\Rightarrow \quad c^1(2\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}) + c^2(-\underset{\sim}{i} + \underset{\sim}{k}) + c^3(4\underset{\sim}{i} - \underset{\sim}{j} + 6\underset{\sim}{k}) = \underset{\sim}{0}$$

$$\Rightarrow \quad 2c^1 - c^2 + 4c^3 = 0$$

$$-c^1 - c^3 = 0 \quad \Rightarrow \quad c^3 = -c^1$$

$$-c^1 + c^2 + 6c^3 = 0$$

$$-2c^1 - c^2 = 0$$

$$-7c^1 + c^2 = 0$$

$$-9c^1 = 0 \Rightarrow c^2 = c^3 = 0. \quad \square$$

(b) $(\underset{\sim}{g}_1 \times \underset{\sim}{g}_2) \cdot \underset{\sim}{g}_3 = -1/9$

$$\underset{\sim}{g}_1 \times \underset{\sim}{g}_2 = -(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}) \Rightarrow \underset{\sim}{g}^3 = \tfrac{1}{9}(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k})$$

$$\underset{\sim}{g}_2 \times \underset{\sim}{g}_3 = \underset{\sim}{i} + 10\underset{\sim}{j} + \underset{\sim}{k} \Rightarrow \underset{\sim}{g}^2 = \tfrac{1}{9}(-7\underset{\sim}{i} - 16\underset{\sim}{j} + 2\underset{\sim}{k})$$

$$\underset{\sim}{g}_3 \times \underset{\sim}{g}_1 = 7\underset{\sim}{i} + 16\underset{\sim}{j} - 2\underset{\sim}{k} \Rightarrow \underset{\sim}{g}^1 = -\tfrac{1}{9}(\underset{\sim}{i} + 10\underset{\sim}{j} + \underset{\sim}{k})$$

$$\boxed{\begin{array}{l} V_1 = \underset{\sim}{g}_1 \cdot \underset{\sim}{V} = 0, \quad V_2 = \underset{\sim}{g}_2 \cdot \underset{\sim}{V} = 0, \quad V_3 = \underset{\sim}{g}_3 \cdot \underset{\sim}{V} = 9 \\ V^1 = \underset{\sim}{g}^1 \cdot \underset{\sim}{V} = -4/3, \quad V^2 = \underset{\sim}{g}^2 \cdot \underset{\sim}{V} = -7/3, \quad V^3 = \underset{\sim}{g}^3 \cdot \underset{\sim}{V} = 1/3 \end{array}}$$

2. (a) We must show that

$$(*) \quad c_1 + c_2 x + c_3 x^2 = 0 \quad \forall\ 0<x<1$$

$\Rightarrow c_1 = c_2 = c_3 = 0$. So it's enough to choose some specific values of x, say, $x = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$. Hence,

$$c_1 + \frac{c_2}{4} + \frac{c_3}{16} = 0 \Rightarrow ①\ 16c_1 + 4c_2 + c_3 = 0$$

$$c_1 + \frac{c_2}{2} + \frac{c_3}{4} = 0 \Rightarrow ②\ 16c_1 + 8c_2 + 4c_3 = 0$$

$$c_1 + \frac{3c_2}{4} + \frac{9c_3}{16} = 0 \Rightarrow ③\ 16c_1 + 12c_2 + 9c_3 = 0$$

$$\begin{array}{ll} ②-① & 4c_2 + 3c_3 = 0 \\ ③-② & 4c_2 + 5c_3 = 0 \end{array} \qquad \det\begin{pmatrix} 4 & 3 \\ 4 & 5 \end{pmatrix} = 20-12 \neq 0$$

$$\Rightarrow c_2 = c_3 = 0$$

$$① \Rightarrow c_1 = 0.$$

Perhaps an easier method is to take advantage of the fact that $(*)$ is identically true for all $0<x<1$, which allows us to differentiate:

$$(*)' \quad c_2 + 2c_3 x = 0 \quad \forall\ 0<x<1$$

$$(*)'' \quad 2c_3 = 0 \Rightarrow c_2 = 0 \overset{(*)}{\Rightarrow} c_1 = 0. \ \square$$

2(b) Define $\underset{\sim}{g}^i = a_0^i + a_1^i x + a_2^i x^2$, $i = 1,2,3$.

For each $i = 1,2,3$, $\underset{\sim}{g}^i \cdot \underset{\sim}{g}_j = \delta^i_j$

generates 3 eq'ns in 3 unknown – solve brute-force $\Rightarrow$

$$\underline{g}^1(x) = 9 - 36x + 30x^2$$

$$\underline{g}^2(x) = -36 + 192x - 180x^2$$

$$\underline{g}^3(x) = 30 - 180x + 180x^2$$