Homework (10) Solutions

(1) $f_{y'} = \dfrac{x y'}{\sqrt{1+(y')^2}}$

$$f_{y'y'} = \frac{x}{\sqrt{1+(y')^2}} - \frac{x(y')^2}{[1+(y')^2]^{3/2}} = \frac{x}{[1+(y')^2]^{3/2}}$$

$f_{y'y'} < 0 \quad \forall\ x \in [-1,0) \Rightarrow$

Legendre's necessary condition not satisfied on $[-1,1]$.

(2) From pp 173-174 (with $\lambda = -T$), we find

$$\delta^2 I(0) = \int_0^1 [(\eta')^2 + T\eta^2]\,dx$$

$$= \int_0^1 (\eta')^2 dx + T\int_0^1 \eta^2 dx$$

$$\geq \int_0^1 (\eta')^2 dx \qquad T \geq 0$$

$$\geq 0 \quad \forall\ \eta \in \mathcal{X}_0$$

and "$= 0$" $\Leftrightarrow \eta = 0$ (shown on p. 163)

Or check for conjugate points:

$$\eta'' - T\eta = 0$$

$\eta(0) = \eta(c) = 0$ any $c \in (0,1]$.

(i) $T = 0 \Rightarrow \eta = Ax + B \Rightarrow \eta = 0$.

(ii) $T > 0 \Rightarrow \eta = A\cosh(\sqrt{T}x) + B\sinh(\sqrt{T}x)$

$\eta(0) = A = 0$

$\eta(c) = B\sinh(\sqrt{T}c) = 0 \Leftrightarrow B = 0$

(since $c, T > 0$)

$\Rightarrow$ no conjugate points $c \in (0,1] \Rightarrow$

$\underline{\delta^2 I(0) \text{ pos. definite}}$.

(3) Let y_0 be a smooth solution of E-L.

p.165

$$\delta^2 I(y_0) = \int_a^b \underbrace{f_{y'y'}(x, y_0(x), y_0'(x))}_{P^0(x) > 0 \text{ (by assumption)}} (\eta')^2 \, dx$$

Check for conjugate points:

$$\frac{d}{dx}\left(P^0(x)\, u'\right) = 0$$

$$u(a) = u(c) = 0 \qquad c \in (a,b]$$

$\Rightarrow P^0(x)\, u' = C_1$

$$\Rightarrow u = C_1 \int_a^x \frac{d\tau}{P^0(\tau)} + C_2$$

$u(a) = C_2 = 0$

$$u(c) = C_1 \int_a^c \frac{d\tau}{P^0(\tau)} = 0$$

Now $P^{\circ}(x) > 0$ on $[a, b]$

$$\Rightarrow \frac{1}{P^{\circ}(x)} > 0 \quad " \quad "$$

$$\Rightarrow \int_a^c \frac{d\tau}{P^{\circ}(\tau)} > 0 \qquad (c > a)$$

$\Rightarrow C_1 = 0 \Rightarrow$ no conjugate points.