Homework (11) Solutions

(1) (a) $y = a\left[1 - \cos\left(\frac{\pi x}{2}\right)\right]$

$y' = \frac{\pi a}{2}\sin\left(\frac{\pi x}{2}\right)$

$y'' = \frac{\pi^2}{4} a \cos\left(\frac{\pi x}{2}\right)$

$$\hat{I}(a) = \int_0^1 \left\{\frac{1}{2}\left(\frac{\pi^2 a}{4}\right)^2 \cos^2\left(\frac{\pi x}{2}\right) - q\,a\left[1 - \cos\left(\frac{\pi x}{2}\right)\right]\right\} dx$$

$$= \frac{1}{2}\,\frac{\pi^4 a^2}{32} - qa + \frac{2qa}{\pi}$$

$$\hat{I}'(a) = \frac{\pi^4 a}{32} + q\left(\frac{2}{\pi} - 1\right) = 0$$

$$\Rightarrow \quad a = \left(1 - \frac{2}{\pi}\right)\frac{32}{\pi^4} q$$

$$\hat{y}(1) = \left(1 - \frac{2}{\pi}\right)\frac{32}{\pi^4} q = 0.119\,q$$

$$y_{ex}(1) = \left(\frac{1}{24} - \frac{1}{6} + \frac{1}{4}\right) q = \frac{3q}{24} = 0.125\,q$$

(2) (a) $\Delta w = -2 a_{11} \pi^2 \sin(\pi x)\sin(\pi y)$

$$\therefore \quad \hat{I}(a_{11}) = \int_0^1 \int_0^1 \frac{D}{2} 2^4 a_{11}^2 \pi^4 \sin^2(\pi x)\sin^2(\pi y)\, dx\, dy$$

$$- \hat{p} \int_0^1 \int_0^1 a_{11} \sin(\pi x)\sin(\pi y)\, dx\, dy$$

$$= \frac{D}{2}\pi^4 a_{11}^2 - \frac{4\hat{p}}{\pi^2} a_{11}$$

Homework (11) sol'n cont.

$$\hat{I}'(a_{11}) = D\pi^4 a_{11} - \frac{4\hat{p}}{\pi^2} = 0$$

$$\Rightarrow \boxed{a_{11} = \frac{4\hat{p}}{D\pi^6}}$$

(b) $$\Delta w = -\sum\sum a_{nm}\left[(n\pi)^2+(m\pi)^2\right]\sin(n\pi x)\sin(m\pi y)$$

$$\int_0^1\int_0^1 (\Delta w)^2\,dx\,dy$$

$$\int_0^1\int_0^1 \sum\sum a_{nm}^2\left[(n\pi)^2+(m\pi)^2\right]^2 \sin^2(n\pi x)\sin^2(m\pi y)\,dx\,dy$$

because all cross product terms integrate to zero, e.g.

$$\int_0^1\int_0^1 \sin(n\pi x)\sin(k\pi x)\sin(m\pi y)\sin(l\pi y)\,dx\,dy$$

$$= 0 \quad n\neq k \text{ and/or } m\neq l$$

$$\rightarrow \sum\sum a_{nm}^2\,\pi^4(n^2+m^2)^2\,\frac{1}{4}$$

$$\int_0^1\int_0^1 \sin n\pi x \sin m\pi y = +\frac{1}{nm\pi^2}\cos n\pi x\Big|_0^1 \cos m\pi y\Big|_0^1$$

$$= \begin{cases} \dfrac{4}{nm\pi^2} & n,m = 1,3,5,\ldots \\ 0 & n,m = 2,4,\ldots \end{cases}$$

Homework (11) Solns cont

$$\hat{I}(a_{11}, a_{13}, a_{31}, a_{33}, \ldots)$$

$$= \frac{1}{8} D \pi^4 \sum_{\substack{n=1 \\ \text{odd}}}^{N} \sum_{\substack{m=1 \\ \text{odd}}}^{N} a_{nm}^2 \left(n^2 + m^2\right)^2 - \hat{p} \frac{4}{\pi^2} \sum_{\substack{n=1 \\ \text{odd}}}^{N} \sum_{\substack{m=1 \\ \text{odd}}}^{N} a_{nm} \frac{1}{nm}$$

$$\frac{\partial \hat{I}}{\partial a_{nm}} = \frac{1}{4} D \pi^4 a_{nm} \left(n^2 + m^2\right)^2 - \hat{p} \frac{4}{\pi^2} \frac{1}{nm} = 0$$

$$\Rightarrow \boxed{a_{nm} = \frac{16 \hat{p}}{D \pi^6} \frac{1}{nm \left(n^2 + m^2\right)^2} \qquad n = 1, 3, 5, \ldots}$$

$$\boxed{a_{nm} = 0 \qquad n \text{ or } m \text{ even}}$$