(1) Consider the wave equation

$$u_{tt} = 9u_{xx} \quad -\infty < x < \infty$$
$$t \geq 0$$

$$u(x,0) = \begin{cases} 2\sin x & 0 \leq x \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

$$u_t(x,0) = 0$$

(a) Find the smallest $T_1 > 0$ such that $u(0, T_1) = 0$.

(b) Find T_2 and $u(5\pi, T_2)$ such that

$$\max_{t>0} u(5\pi, t) = u(5\pi, T_2).$$

(2) Consider the damped wave equation

$$(*) \quad u_{tt} + \gamma u_t = c^2 u_{xx} \quad (\gamma > 0).$$

(a) Show that $\frac{dE}{dt} \leq 0$ on all solutions.

(b) Use this to show uniqueness of solution for $(*)$ subj. to $u(x,0) = \hat{\phi}(x)$, $u_t = \hat{\psi}(x)$ (assuming that both $\hat\phi$ & $\hat\psi$ vanish for all $|x| > R > 0$).

(3) Hamilton's Principle for the elastic rod:

Consider a semi-infinite 1-d elastic rod.

Lagrangian density

$$L \equiv \underbrace{\int_0^\infty \frac{1}{2} u_t^2 \, dx}_{T \text{ (K.E.)}} - \underbrace{\int_0^\infty \frac{c^2}{2} u_x^2 \, dx}_{V \text{ (P.E.)}}$$

(We assume that any initial conditions vanish for all $x > R > 0 \Rightarrow$ the above integrals converge.)

Then

$$I(u) = \int_{t_1}^{t_2} L \, dt = \frac{1}{2}\int_{t_1}^{t_2}\int_0^\infty (u_t^2 - c^2 u_x^2) \, dx \, dt$$

Assume $\lim_{x\to\infty} u(x,t) = 0$ and work with

$$I_0 = \left\{ \eta \in C^1([0,\infty)\times[0,\infty)) : \eta(x,t_1) = \eta(x,t_2) = 0, \ \lim_{x\to\infty} \eta(x,t) = 0 \right\}$$

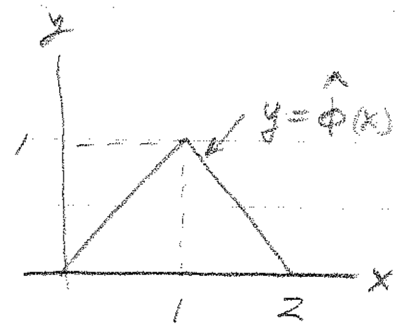
Use Hamilton's principle ($\delta I = 0 \ \forall \eta \in I_0$) to deduce:

(a) The governing pde (E-L)
(b) The natural b.c. at $x = 0$.

(4) Consider a semi-infinite rod with a fixed end:

$$\left.\begin{aligned} u_{tt} &= 4u_{xx} \qquad 0 < x < \infty \\ u(0,t) &= 0 \end{aligned}\right\} \ t > 0$$

Subj. to the I.C.'s:



$$u(x,0) = \begin{cases} x & 0 \le x \le 1 \\ 2-x & 1 \le x < 2 \\ 0 & x \ge 2 \end{cases}$$

$$u_t(x,0) = 0.$$

(a) Recall that the stress $\sigma = E u_x$. Taking $E=1$, (a) calculate the maximum and minimum stress at $x=0$, i.e.,

$$\max_{t \ge 0} u_x(0,t), \quad \min_{t \ge 0} u_x(0,t), \text{ respectively.}$$

(b) At what time(s) ($t = ?$) do these occur?