Homework (13) Solutions

(1) From p. 219 (notes)

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \hat{\psi}_{ext}(\tau)\,d\tau,$$

where $\hat{\psi}_{ext}(x)$ is odd & 2ℓ-periodic. Hence,

$$\hat{\psi}_{ext}(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{\ell},$$

where $a_n = \frac{2}{\ell}\int_0^{\ell} \hat{\psi}(x)\sin\frac{n\pi x}{\ell}\,dx$

$$u(x,t) = \frac{1}{2c}\sum_{n=1}^{\infty}\frac{a_n \ell}{n\pi}\left\{\cos\left[\frac{n\pi}{\ell}(x-ct)\right] - \cos\left[\frac{n\pi}{\ell}(x+ct)\right]\right\}$$

$$= \frac{\ell}{\pi c}\sum_{n=1}^{\infty}\frac{a_n}{n}\sin\left(\frac{n\pi x}{\ell}\right)\sin\left(\frac{n\pi c t}{\ell}\right)$$

By separation of variables (p. 222),

$$u(x,t) = \sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{\ell}\right)\left(A_n\cos\left(\frac{n\pi c t}{\ell}\right) + B_n\sin\left(\frac{n\pi c t}{\ell}\right)\right)$$

$$0 = u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right) \Rightarrow A_n = 0 \quad n=1,2,\ldots$$

$$\hat{\psi}(x) = u_t(x,0) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{\ell}\sin\left(\frac{n\pi x}{\ell}\right)$$

$$\Rightarrow \quad B_n \frac{n\pi c}{l} = \frac{2}{l}\int_0^l \psi(x) \sin\frac{n\pi x}{l}\,dx = a_n$$

Fourier coefficient

$$\Rightarrow \quad B_n = \frac{a_n l}{n\pi c}$$

$$\Rightarrow u(x,t) = \frac{l}{\pi c}\sum_{n=1}^{\infty} \frac{a_n}{n} \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi c t}{l}\right).$$

(same ✓)

② (a) max occurs on boundary $(r=2)$:

$$\frac{\partial u(2,\theta)}{\partial\theta} = 6\cos 2\theta = 0 \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

$$\boxed{\max_{D} u = 4} \quad \left(@\ \theta = \frac{\pi}{4}, \frac{3\pi}{4}\right)$$

(b) $$u(0,0) = \frac{1}{2\pi\cancel{(2)}}\int_0^{2\pi} (1 + 3\sin 2\theta)\cancel{2}\,d\theta$$

(mean-value property)

$$= \frac{1}{2\pi}\left(2\pi - \frac{3}{2}\cos 2\theta\Big|_0^{2\pi}\right)$$

$$\boxed{u(0,0) = 1}$$

③ Separate variables $u(r,\theta) = R(r)\,\Theta(\theta)$

$$\Rightarrow \quad \ddot{\Theta} + \lambda\Theta = 0 \qquad \Theta(0) = \Theta(\pi) = 0$$

$$\Rightarrow \left.\begin{array}{l}\lambda_n = n^2 \\ \Theta_n = \sin(n\theta)\end{array}\right\} \quad n = 1, 2, \ldots$$

$R_n(r) = r^n$ (p. 234 notes)

$\Rightarrow \quad u(r,\theta) = \sum_{n=1}^{\infty} A_n r^n \sin(n\theta)$ gen. sol'n.

b.c. $u(1,\theta) = \pi \sin\theta - \sin 2\theta \overset{?}{=} \sum_{n=1}^{\infty} A_n \sin(n\theta)$

$\Rightarrow A_1 = \pi, \quad A_2 = -1,$ all other $A_n = 0$.

$\Rightarrow \boxed{u(r,\theta) = \pi r \sin\theta - r^2 \sin 2\theta}$.