Homework (14) Solutions

(1)
$$u(x,t) = \frac{1}{2\alpha\sqrt{\pi t}}\int_{0}^{\infty} \exp\left[\frac{-(x-y)^2}{4\alpha^2 t}\right] dy$$

$$+ \frac{3}{2\alpha\sqrt{\pi t}}\int_{-\infty}^{0} \exp\left[\frac{-(x-y)^2}{4\alpha^2 t}\right] dy$$

1st integral: $\frac{y-x}{2\alpha\sqrt{t}} = p$

2nd integral: $\frac{x-y}{2\alpha\sqrt{t}} = q$

$$u(x,t) = \frac{1}{2}\left\{\frac{2}{\sqrt{\pi}}\int_{\frac{-x}{2\alpha\sqrt{t}}}^{\infty} e^{-p^2}dp + 3\cdot\frac{2}{\sqrt{\pi}}\int_{\frac{x}{2\alpha\sqrt{t}}}^{\infty} e^{-q^2}dq\right\}$$

$$= \frac{1}{2}\left\{\frac{2}{\sqrt{\pi}}\int_{\frac{-x}{2\alpha\sqrt{t}}}^{0} e^{-p^2}dp + 1\right.$$

$$\left. + 3\left[1 - \frac{2}{\sqrt{\pi}}\int_{0}^{\frac{x}{2\alpha\sqrt{t}}} e^{-q^2}dq\right]\right\}$$

$$= \frac{1}{2}\left\{\operatorname{erf}\left(\frac{x}{2\alpha\sqrt{t}}\right) + 4 - 3\operatorname{erf}\left(\frac{x}{2\alpha\sqrt{t}}\right)\right]$$

$$\boxed{u(x,t) = 2 - \operatorname{erf}\left(\frac{x}{2\alpha\sqrt{t}}\right)}$$

(2) (a) Define $\hat{\phi}_{ev}(x) = \begin{cases}\hat{\phi}(x) & 0 < x < \infty \\ \hat{\phi}(-x) & -\infty < x < 0\end{cases}$

$$\therefore\ u(x,t) = \frac{1}{2\alpha\sqrt{\pi t}}\int_{-\infty}^{\infty} \exp\left[\frac{-(x-y)^2}{4\alpha^2 t}\right]\hat{\phi}_{ev}(y)\,dy$$

(b) $$u(-x,t) = \frac{1}{2\alpha\sqrt{\pi t}} \int_{-\infty}^{\infty} \exp\left[\frac{-(x+y)^2}{4\alpha^2 t}\right] \hat{\phi}_{ev}(y)\, dy$$

$$y = -\tau \quad dy = -d\tau$$

$$= \frac{1}{2\alpha\sqrt{\pi t}} \int_{-\infty}^{\infty} \exp\left[\frac{-(x-\tau)^2}{4\alpha^2 t}\right] \underbrace{\hat{\phi}_{ev}(-\tau)}_{\hat{\phi}_{ev}(\tau)}\, d\tau$$

$$u(-x,t) = u(x,t) \quad \text{even} \checkmark$$

(c) $$u(x,t) = \frac{1}{2\alpha\sqrt{\pi t}} \int_{0}^{\infty} \left\{ \exp\left[\frac{-(x-y)^2}{4\alpha^2 t}\right] + \exp\left[\frac{-(x+y)^2}{4\alpha^2 t}\right] \right\} dy$$

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$$= \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x}{2\alpha\sqrt{t}}\right) - \left(\operatorname{erf}\left(\frac{x}{2\alpha\sqrt{t}}\right) - 1\right)\right]$$

(sign changes)

$$\boxed{u(x,t) \equiv 1}$$

(Remark: We could have guessed this from the start!)

(3) Cf. p. 261: same argument works because the boundary term $\alpha w_x w \big|_0^l$ vanishes because $w_x\big|_0 = w_x\big|_l = 0$.

(4) (a) $\tilde{I}(y,\lambda) = \int_0^1 [(y')^2 - \lambda y^2]\, dx$

$$\delta \tilde{I} = \int_0^1 (2y'\eta' - 2\lambda y \eta)\, dx = 0 \quad \forall \eta \in C^1([0,1]).$$

$$\overset{\text{parts}}{=} -\int_0^1 2(y'' + \lambda y)\eta\, dx - y'\eta\Big|_0^1 = 0 \quad \forall \eta$$

Suppose first that $\eta(0) = \eta(1) = 0$, but otherwise arbitrary. Then

E-L $\boxed{y'' + \lambda y = 0}$.

Next let $\eta(0)$ & $\eta(1)$ be arbitrary $\Rightarrow$

$$\boxed{y'(0) = y'(1) = 0}$$

(b) $\boxed{\lambda = 0}$: $y'' = 0 \Rightarrow y = Ax + B$

$y'(0) = 0 \Rightarrow A = 0$ B-arbitrary const

$\boxed{y \equiv 1}$ eigenfunction

Also $\left.\begin{array}{l} \lambda = n^2 \\ y_n = \cos(nx) \end{array}\right\}$ $n = 1, 2, \ldots$

(5) $u(x,t) = X(x)T(t) \Rightarrow$

$X'' + \lambda X = 0 \quad X'(0) = X'(1) = 0$

(Solutions given above)

$\dot{T}_n = -\alpha^2 \lambda_n T_n \Rightarrow T_n = e^{-\alpha^2 n^2 t}$ $n = 1, 2, \ldots$

$T_0 = 1$

Gen. solution $u(x,t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(nx) e^{-\alpha^2 n^2 t}$

$$u(x,0) = 8\cos^2 x = 8\left(\frac{1+\cos 2x}{2}\right)$$

$$= 4 + 4\cos 2x$$

$\Rightarrow$ $C_0 = 4$, $C_2 = 4$ all other C_n's $= 0$.

$$\Rightarrow \boxed{u(x,t) = 4 + 4\cos 2x\, e^{-4\alpha^2 t}}$$