Homework ⑤ Solutions

① $\Delta u = g^{ij} \left(\frac{\partial u}{\partial x^j}\right)\Big|_i$ (p. 63)

$$[g^{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/(x^1)^2 & 0 \\ 0 & 0 & 1/(x^1 \sin(x^2))^2 \end{bmatrix}$$ (Homework ④)

p. 47 $\begin{cases} \underset{\sim}{g}_1 = \sin(x^2)\, \underset{\sim}{e}_r(x^3) + \cos(x^2)\, \underset{\sim}{e}_3 \\ \underset{\sim}{g}_2 = x^1 [\cos(x^2)\, \underset{\sim}{e}_r(x^3) - \sin(x^2)\, \underset{\sim}{e}_3] \\ \underset{\sim}{g}_3 = x^1 \sin(x^2)\, \underset{\sim}{e}_\theta(x^3) \end{cases}$,

where $\begin{cases} \underset{\sim}{e}_r(x^3) = \cos(x^3)\, \underset{\sim}{e}_1 + \sin(x^3)\, \underset{\sim}{e}_2 = -\underset{\sim}{e}_\theta'(x^3) \\ \underset{\sim}{e}_\theta(x^3) = -\sin(x^3)\, \underset{\sim}{e}_1 + \cos(x^3)\, \underset{\sim}{e}_2 = \underset{\sim}{e}_r'(x^3). \end{cases}$

$\frac{\partial \underset{\sim}{g}_1}{\partial x^1} = \underset{\sim}{0}$, $\frac{\partial \underset{\sim}{g}_2}{\partial x^1} = \frac{1}{x^1} \underset{\sim}{g}_2$, $\frac{\partial \underset{\sim}{g}_3}{\partial x^1} = \frac{1}{x^1} \underset{\sim}{g}_3$

$\frac{\partial \underset{\sim}{g}_1}{\partial x^2} = \frac{1}{x^1} \underset{\sim}{g}_2$, $\frac{\partial \underset{\sim}{g}_2}{\partial x^2} = -x_1 \underset{\sim}{g}_1$, $\frac{\partial \underset{\sim}{g}_3}{\partial x^2} = \cot(x^2)\, \underset{\sim}{g}_3$

$\frac{\partial \underset{\sim}{g}_1}{\partial x^3} = \frac{1}{x^1} \underset{\sim}{g}_3$, $\frac{\partial \underset{\sim}{g}_2}{\partial x^3} = \cot(x^2) \underset{\sim}{g}_3$, $\frac{\partial \underset{\sim}{g}_3}{\partial x^3} = -x^1 \sin(x^2)\, \underset{\sim}{e}_r(\theta)$

$$\left\{ {i \atop k\,i} \right\} \equiv \underset{\sim}{g}^i \cdot \frac{\partial \underset{\sim}{g}_k}{\partial x^i} = \underset{\sim}{g}^i \cdot \frac{\partial \underset{\sim}{g}_i}{\partial x^k}$$

$$\left\{ {2 \atop 2\,1} \right\} = \left\{ {2 \atop 1\,2} \right\} = \frac{1}{x^1} = \left\{ {3 \atop 3\,1} \right\} = \left\{ {3 \atop 1\,3} \right\}$$

$$\left\{ {3 \atop 3\,2} \right\} = \left\{ {3 \atop 2\,3} \right\} = \cot(x^2)$$

$$\left(g^{ij}\frac{\partial u}{\partial x^j}\right) = \begin{pmatrix} \frac{\partial u}{\partial x^1} \\ \frac{1}{(x^1)^2}\frac{\partial u}{\partial x^2} \\ \frac{1}{[(x^1)\sin(x^2)]^2}\frac{\partial u}{\partial x^3} \end{pmatrix} \equiv \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix}$$

$$v^i|_i = \frac{\partial v^i}{\partial x^i} + \left\{ {i \atop ki} \right\} v^k$$

$$\therefore \Delta u = \frac{\partial^2 u}{\partial x^{1\,2}} + \frac{1}{(x^1)^2}\frac{\partial^2 u}{\partial x^{2\,2}} + \frac{1}{(x^1)^2\sin^2(x^2)}\frac{\partial^2 u}{\partial x^{3\,2}}$$

$$+ \overset{\frac{1}{x^1}}{\left\{ {2 \atop 12} \right\}} \frac{\partial u}{\partial x^1} + \overset{1/x^1}{\left\{ {3 \atop 13} \right\}} \frac{\partial u}{\partial x^1}$$

$$+ \overset{\cot(x^2)}{\left\{ {3 \atop 23} \right\}} \frac{1}{(x^1)^2}\frac{\partial u}{\partial x^2}$$

$$x^1 = \rho\,, \quad x^2 = \phi\,, \quad x^3 = \theta$$

$$\boxed{\Delta u = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2} + \frac{2}{r}\frac{\partial u}{\partial \rho} + \frac{\cot\phi}{r^2}\frac{\partial u}{\partial \phi}}$$

see (1) above

(2) $\underset{\sim}{r} = (a + b\cos(x^1))\, \underset{\sim}{e}_r(x^2) + b\sin(x^1)\, \underline{e}_3$

$$\Rightarrow \begin{cases} \underset{\sim}{g}_1 = -b\sin(x^1)\, \underline{e}_r(x^2) + b\cos(x^1)\, \underline{e}_3 \\ \underline{g}_2 = (a + b\cos(x^1))\, \underset{\sim}{e}_\theta(x^2) \end{cases}$$

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$$\therefore \quad [g_{\alpha\beta}] = \begin{bmatrix} b^2 & 0 \\ 0 & (a+b\cos(x^1))^2 \end{bmatrix}$$

$$\Rightarrow \quad \sqrt{g} = b\left(a+b\cos(x^1)\right)$$

$$A = \int_0^{2\pi}\int_0^{2\pi} b\left(a+b\cos(x^1)\right) dx^1\, dx^2$$

$$= ab(2\pi)^2 + 2\pi b^2 \cancel{\sin(x^1)\Big|_0^{2\pi}}^{\,0}$$

$$\boxed{A = 4\pi^2 ab}$$

③ (a) $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + (y^2-x^2)\underset{\sim}{k}$

$(x=x^1)$ $\underset{\sim}{g}_1 = \underset{\sim}{i} - 2x\underset{\sim}{k}$

$(y=x^2)$ $\underset{\sim}{g}_2 = \underset{\sim}{j} + 2y\underset{\sim}{k}$

$\underset{\sim}{g}_1 \times \underset{\sim}{g}_2 = 2x\underset{\sim}{i} - 2y\underset{\sim}{j} + \underset{\sim}{k}$

$$\Rightarrow \boxed{\underset{\sim}{n}(\underset{\sim}{r}) = \frac{1}{\sqrt{x^2+y^2+\frac{1}{4}}}\left(x\underset{\sim}{i} - y\underset{\sim}{j} + \frac{1}{2}\underset{\sim}{k}\right)}$$

(b) $$\frac{\partial n}{\partial x^1} = \underset{\sim}{\nabla}\underset{\sim}{n}\,\underset{\sim}{i} = \frac{1}{\sqrt{x^2+y^2+\frac{1}{4}}}\underset{\sim}{i} - \frac{x}{(x^2+y^2+\frac{1}{4})^{3/2}}\left(x\underset{\sim}{i} - y\underset{\sim}{j} + \frac{1}{2}\underset{\sim}{k}\right)$$

$$\Rightarrow \underset{\sim}{\nabla}\underset{\sim}{n}(\underset{\sim}{0})\,\underset{\sim}{i} = (\quad)\Big|_{x=y=0} = \frac{1}{\sqrt{1/4}}\underset{\sim}{i} = 2\underset{\sim}{i} \quad ✓$$

similarly $$\frac{\partial \underset{\sim}{n}}{\partial x^2} = \frac{-1}{\sqrt{x^2+y^2+1/4}}\underset{\sim}{j} + \ldots$$

$\Rightarrow \quad \underset{\sim}{\nabla} \underset{\sim}{n}(\underset{\sim}{o}) \underset{\sim}{j} = -2 \underset{\sim}{j}$ ✓

(c) $\underset{\sim}{B}(\underset{\sim}{o}) = - \underset{\sim}{\nabla} \underset{\sim}{n}(\underset{\sim}{o})$

$$\underset{\sim}{B}(\underset{\sim}{o}) \underset{\sim}{i} = - \underset{\sim}{\nabla} \underset{\sim}{n}(\underset{\sim}{o}) \underset{\sim}{i} = -2 \underset{\sim}{i}$$

$$\underset{\sim}{B}(\underset{\sim}{o}) \underset{\sim}{j} = - \underset{\sim}{\nabla} \underset{\sim}{n}(\underset{\sim}{o}) \underset{\sim}{j} = 2 \underset{\sim}{j}$$

$\Rightarrow \quad K_1 = -2$ prin. dir. $\underset{\sim}{i}$

$\quad K_2 = 2$ " " $\underset{\sim}{j}$.

Indeed: