Homework ⑥ Solutions

(1) $x^1 = x$, $x^2 = y$:

$$\underset{\sim}{g}_1 = \underset{\sim}{e}_1 + h_x \underset{\sim}{e}_3$$

$$\underset{\sim}{g}_2 = \underset{\sim}{e}_2 + h_y \underset{\sim}{e}_3$$

$$[g_{\alpha\beta}] = \begin{bmatrix} 1+h_x^2 & h_x h_y \\ h_x h_y & 1+h_y^2 \end{bmatrix}$$

$$g = \det[g_{\alpha\beta}] = 1 + h_x^2 + h_y^2$$

$$\underset{\sim}{n} = \frac{1}{\sqrt{g}}(\underset{\sim}{g}_1 \times \underset{\sim}{g}_2) = \frac{1}{\sqrt{g}}(-h_x \underset{\sim}{e}_1 - h_y \underset{\sim}{e}_2 + \underset{\sim}{e}_3)$$

$$\underset{\sim}{B} = -\nabla \underset{\sim}{n}$$

(p.76)

$$B_{\alpha\beta} = -\underset{\sim}{g}_\alpha \cdot \nabla \underset{\sim}{n}\, \underset{\sim}{g}_\beta = -\underset{\sim}{g}_\alpha \cdot \frac{\partial \underset{\sim}{n}}{\partial x^\beta} = \underset{\sim}{n} \cdot \frac{\partial \underset{\sim}{g}_\beta}{\partial x^\alpha}$$

$$\frac{\partial \underset{\sim}{g}_1}{\partial x^1} = h_{xx} \underset{\sim}{e}_3 \qquad \frac{\partial \underset{\sim}{g}_1}{\partial x^2} = h_{xy} \underset{\sim}{e}_3$$

$$\frac{\partial \underset{\sim}{g}_2}{\partial x^1} = h_{yx} \underset{\sim}{e}_3 \qquad \frac{\partial \underset{\sim}{g}_2}{\partial x^2} = h_{yy} \underset{\sim}{e}_3$$

$$\Rightarrow [B_{\alpha\beta}] = \frac{1}{\sqrt{1+h_x^2+h_y^2}} \begin{bmatrix} h_{xx} & h_{xy} \\ h_{xy} & h_{yy} \end{bmatrix}$$

p.83: $$H = \frac{1}{2g}\left(B_{11} g_{22} - 2B_{12} g_{12} + B_{22} g_{11}\right)$$

$$\boxed{\therefore H = \frac{(1+h_y^2)h_{xx} - 2h_x h_y h_{xy} + (1+h_x^2)h_{yy}}{2(1+h_x^2+h_y^2)^{3/2}}}$$

p. 85 $\quad K = \dfrac{\det[B_{\alpha\beta}]}{g}$

$$\boxed{\therefore K = \frac{h_{xx}h_{yy} - h_{xy}^2}{(1 + h_x^2 + h_y^2)^2}}$$

2. (a) $\underset{\sim}{r}' = a[-\sin(\theta)\underset{\sim}{e}_1 + \cos\theta\,\underset{\sim}{e}_2] + z'(\theta)\,\underset{\sim}{e}_3$

$\therefore \quad \underset{\sim}{r}' \cdot \underset{\sim}{r}' = a^2 + (z')^2$

Arc-length functional:

$$I(r) = \int_{\theta_1}^{\theta_2} \sqrt{\underset{\sim}{r}' \cdot \underset{\sim}{r}'}\, d\theta$$

$$= \int_{\theta_1}^{\theta_2} \sqrt{a^2 + (z')^2}\, d\theta$$

$$z(\theta_1) = z_1, \quad z(\theta_2) = z_2$$

<u>E-L</u>: $\dfrac{z'}{\sqrt{a^2 + (z')^2}} = C$

$$\Rightarrow z' = \tilde{C}$$

$$z = \tilde{C}\theta + D$$

$$z_1 = \tilde{C}\theta_1 + D$$

$$z_2 = \tilde{C}\theta_2 + D$$

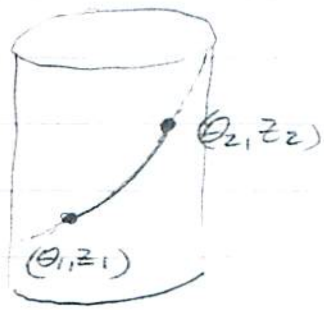
$\Rightarrow \tilde{C} = \dfrac{z_2 - z_1}{\theta_2 - \theta_1}$ (assuming $\theta_1 \neq \theta_2$)

$$D = \frac{z_1\theta_2 - z_2\theta_1}{\theta_2 - \theta_1}$$

(b) In general,

$$\tilde{\underset{\sim}{r}}(\theta) = a\,\underset{\sim}{e}_r(\theta) + (\hat{C}\theta + D)\underset{\sim}{e}_3 \ , \quad \theta_1 \le \theta \le \theta_2 ,$$

is an arc of a helix.



If $z_1 = z_2$, then the helix becomes an arc of a circle. If $\theta_1 = \theta_2$, our formulation fails (we need to write $\theta(z)$). Nonetheless, it's not hard to show ~~that~~ the stationary curve is a verticle line.

(3) E-L:

(a) $$\begin{cases} 2y' + 3(y')^2 = C \\ y(0) = y(1) = 0 \end{cases}$$

(b) $\boxed{y \equiv 0 \quad \text{is a solution}}$