Homework (7) Solutions

(1) $\frac{d}{d\alpha} I(y+\alpha\eta)\Big|_{\alpha=0} = \frac{d}{d\alpha}\left\{\frac{1}{2}\int_0^1 (y''+\alpha\eta'')^2 dx - P\left(y(1)+\alpha\eta(1)\right)\right\}\Big|_{\alpha=0}$

(p.113)

$\Rightarrow \delta I = \int_0^1 y''''\eta\, dx + y''\eta'\Big|_0^1 - y'''\eta\Big|_0^1 - P\eta(1) = 0$

$X_0 = \{C^2[0,1] : \eta(0)=\eta'(1)=0\} \quad \forall \eta \in X_0$

$\therefore \delta I = \int_0^1 y''''\eta\, dx - (y'''(1)+P)\eta(1) + y''(0)\eta'(0) = 0 \quad \forall \eta \in X_0$

Choose all $\eta \in X_0$ to satisfy $\eta(1)=\eta'(0)=0$.
Then

E-L : $\boxed{\begin{array}{l} y''''=0 \\ y(0)=y'(1)=0 \end{array}}$ ← geom. b.c.'s

Then abitrary $\begin{cases}\eta(1) \Rightarrow \\ \eta'(0) \Rightarrow\end{cases}$ $\boxed{\begin{array}{l} y'''(1)=-P \\ y''(0)=0 \end{array}}$ natural b.c's

Integrate: $y''' = c_1$

$y'''(1) = -P \Rightarrow \underline{c_1 = -P}$

$y'' = -Px + c_2$

$y''(0) = \underline{c_2 = 0}$

$y' = -\frac{Px^2}{2} + c_3$

$y'(1) = -\frac{P}{2} + c_3 = 0 \Rightarrow c_3 = \frac{P}{2}$

$y = -\frac{Px^3}{6} + \frac{P}{2}x + c_4$

$$y(0) = 0 = c_4$$

$$\Rightarrow \boxed{y = -\frac{Px^3}{6} + \frac{Px}{2}}$$

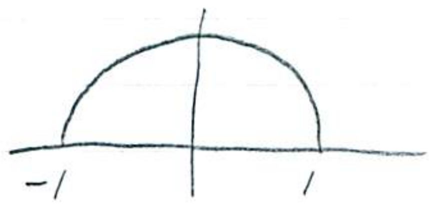
(2) (a) $I^*(y, \lambda) = \int_{-1}^{1} y\,dx - \lambda\left[\int_{-1}^{1}\sqrt{1+(y')^2}\,dx - L\right]$

E-L $\quad 1 + \lambda \frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) = 0$

$$\Rightarrow \frac{y''}{1+(y')^2} - \frac{(y')^2 y''}{(1+(y')^2)^{3/2}} = -\frac{1}{\lambda}$$

$$\Rightarrow \boxed{\frac{y''}{(1+(y')^2)^{3/2}} = -\frac{1}{\lambda}}$$ constant curvature (circle)

(b) $L = \pi \Rightarrow R = 1 = \lambda$



(3) (cf. p. 72) $g_{11} = 1 \quad g_{12} = 0$

(a) $\qquad g_{22} = \sin^2\phi$

$\therefore I(y_1, y_2) = \int_{t_0}^{t_1} \sqrt{(y_1')^2 + (y_2')^2 \sin^2(y_1)}\,dt$

(b) $\frac{\partial f}{\partial y_1} = \frac{(y_2')^2 \sin(y_1)\cos(y_1)}{\sqrt{(y_1')^2 + (y_2')^2 \sin^2(y_1)}}$

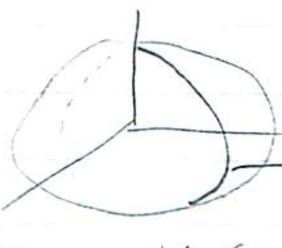
$$\frac{\partial f}{\partial y_2} = 0$$

$$\boxed{\begin{aligned}
&(\text{E-L}) \quad (i) \quad \frac{(y_2')^2 \sin(y_1)\cos(y_1)}{\sqrt{(y_1')^2 + (y_2')^2 \sin^2(y_1)}} - \frac{d}{dx}\left(\frac{y_1'}{\sqrt{(y_1')^2 + (y_2')^2 \sin^2(y_1)}}\right) = 0 \\
&\underline{\text{and}} \\
&(ii) \quad \frac{y_2' \sin^2(y_1)}{\sqrt{(y_1')^2 + (y_2')^2 \sin^2(y_1)}} = C
\end{aligned}}$$

(c) $y_2 \equiv C \Rightarrow y_2' \equiv 0$ $\quad y_1(t)$ arbitrary ($y_1' \neq \vec{0}$)

(i): $0 - \frac{d}{dx}\left(\frac{y_1'}{y_1'}\right) = 0$ ✓

(ii) $C = 0$



"vertical" great circle

$y_1 = \pi/2$ $\quad y_2(t)$ arbitrary

(i) $0 = 0$ $\quad$ (ii) $\frac{y_2'}{y_2'} = C = 1$ ($y_2' \neq 0$)

equator