TAM 611 Homework Set (8) 3/11/09

(due 3/23) 1/2

(1) Given $a : \bar{\Omega} \subset \mathbb{R}^3 \to \mathbb{R}$ and $\underset{\sim}{g} : \bar{\Omega} \to \mathbb{R}^3$ (each of class C^1): (a) show that

$$\underset{\sim}{\nabla}(a\underset{\sim}{g}) = a\underset{\sim}{\nabla}\underset{\sim}{g} + \underset{\sim}{g} \otimes \underset{\sim}{\nabla} a .$$

(b) Hence, deduce that

$$\underset{\sim}{\nabla} \cdot (a\underset{\sim}{g}) = a\,\underset{\sim}{\nabla} \cdot \underset{\sim}{g} + \underset{\sim}{g} \cdot \underset{\sim}{\nabla} a .$$

(2) Consider a generalization of Dirichlet's integral:

$$I(u) = \frac{1}{2}\int_{\Omega} \underset{\sim}{\nabla} u \cdot (\underset{\sim}{K}\,\underset{\sim}{\nabla} u)\, d\underset{\sim}{x}, \quad (\Omega \subset \mathbb{R}^3)$$

where $\underset{\sim}{K} \in Sym(\mathbb{R}^3)$ and K is positive definite. (In heat transfer $\underset{\sim}{K}$ is called the conductivity tensor.)

(a) Determine the (E-L) equation for $I(u)$ — both in direct notation and in components relative to a rectangular Cartesian coordinate system.

(b) If $u|_{\partial\Omega}$ is not prescribed, determine the natural boundary conditions — again, express your results both ways.

③ Determine the (E-L) equation for the surface-area functional (p. 96)

$$I(u) = \iint_{\Omega} \sqrt{1+\left(\frac{\partial u}{\partial x_1}\right)^2+\left(\frac{\partial u}{\partial x_2}\right)^2}\, dx_1\, dx_2$$

$$u\big|_{\partial\Omega} \equiv 0. \qquad (\Omega \subset \mathbb{R}^2)$$