Homework 8 Solutions

1 (a) $\frac{d}{d\alpha}\left[a(\underset{\sim}{x}+\alpha\underset{\sim}{\eta})\,\underset{\sim}{g}(\underset{\sim}{x}+\alpha\underset{\sim}{\eta})\right]\Big|_{\alpha=0}$

$$= \underset{\sim}{\nabla} a\cdot\underset{\sim}{\eta}\,\underset{\sim}{g} + a\,\underset{\sim}{\nabla}\underset{\sim}{g}\,\underset{\sim}{\eta}$$

$$= (\underset{\sim}{g}\otimes\underset{\sim}{\nabla} a + a\,\underset{\sim}{\nabla}\underset{\sim}{g})\underset{\sim}{\eta} \qquad \forall\,\underset{\sim}{\eta}\in\mathbb{R}^3.$$

$$\Rightarrow\quad \underset{\sim}{\nabla}(a\underset{\sim}{g}) = \underset{\sim}{g}\otimes\underset{\sim}{\nabla} a + a\,\underset{\sim}{\nabla}\underset{\sim}{g}$$

(b) $\underset{\sim}{\nabla}\cdot(a\underset{\sim}{g}) = \mathrm{tr}\left(\underset{\sim}{\nabla}(a\underset{\sim}{g})\right) = \underset{\sim}{g}\cdot\underset{\sim}{\nabla} a + a\,\underset{\sim}{\nabla}\cdot\underset{\sim}{g}.$

2 $\delta I = \int_\Omega \underset{\sim}{K}\,\underset{\sim}{\nabla} u\cdot\underset{\sim}{\nabla}\eta\,d\underset{\sim}{x}$

$$= \int_\Omega\left[\underset{\sim}{\nabla}\cdot(\underset{\sim}{K}\,\underset{\sim}{\nabla} u\,\eta) - \underset{\sim}{\nabla}\cdot(\underset{\sim}{K}\,\underset{\sim}{\nabla} u)\eta\right]dx$$

div thm

$$= -\int_\Omega \underset{\sim}{\nabla}\cdot(\underset{\sim}{K}\,\underset{\sim}{\nabla} u)\eta\,d\underset{\sim}{x} + \int_{\partial\Omega}\underset{\sim}{K}\,\underset{\sim}{\nabla} u\cdot\underset{\sim}{n}\,\eta\,dS$$

(a)

<u>E-L</u> $\quad \underset{\sim}{\nabla}\cdot(\underset{\sim}{K}\,\underset{\sim}{\nabla} u) = 0$;

components $\quad \frac{\partial}{\partial x_i}\left(K_{ij}\frac{\partial u}{\partial x_j}\right) = K_{ij}\frac{\partial^2 u}{\partial x_i\partial x_j}$.

(b) $\left(\underset{\sim}{K}\,\underset{\sim}{\nabla} u\cdot\underset{\sim}{n}\right)\Big|_{\partial\Omega} = 0$;

comp. $\quad K_{ij}\frac{\partial u}{\partial x_j}n_i\Big|_{\partial\Omega} = 0$

Homework ⑧ sol'ns cont.

③ $$f = \sqrt{1+\left(\frac{\partial u}{\partial x_1}\right)^2+\left(\frac{\partial u}{\partial x_2}\right)^2}$$

$$\frac{\partial f}{\partial v_1} = \frac{\frac{\partial u}{\partial x_1}}{f} \qquad \frac{\partial f}{\partial v_2} = \frac{\frac{\partial u}{\partial x_2}}{f}$$

$$\frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial v_1}\right) = \frac{\frac{\partial^2 u}{\partial x_1{}^2}}{f} - \frac{\frac{\partial u}{\partial x_1}}{f^2}\,\frac{\left(\frac{\partial u}{\partial x_1}\frac{\partial^2 u}{\partial x_1{}^2}+\frac{\partial u}{\partial x_2}\frac{\partial^2 u}{\partial x_1\partial x_2}\right)}{f}$$

$$= \frac{1}{f^3}\left\{\frac{\partial^2 u}{\partial x_1{}^2}\left[1+\cancel{\left(\frac{\partial u}{\partial x_1}\right)^2}+\left(\frac{\partial u}{\partial x_2}\right)^2\right] - \cancel{\left(\frac{\partial u}{\partial x_1}\right)^2}\frac{\partial^2 u}{\partial x_1{}^2} - \frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2}\frac{\partial^2 u}{\partial x_1\partial x_2}\right\}$$

$$\Rightarrow \frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial v_2}\right) = \frac{1}{f^3}\left\{\frac{\partial^2 u}{\partial x_2{}^2}\left[1+\left(\frac{\partial u}{\partial x_1}\right)^2+\cancel{\left(\frac{\partial u}{\partial x_2}\right)^2}\right] - \cancel{\left(\frac{\partial u}{\partial x_2}\right)^2}\frac{\partial^2 u}{\partial x_2{}^2} - \frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2}\frac{\partial^2 u}{\partial x_1\partial x_2}\right\}$$

E-L (p.134):

$$\cancel{-\frac{1}{f^3}}\left\{\left[1+\left(\frac{\partial u}{\partial x_2}\right)^2\right]\frac{\partial^2 u}{\partial x_1{}^2} - 2\frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2}\frac{\partial^2 u}{\partial x_1\partial x_2} + \left[1+\left(\frac{\partial u}{\partial x_1}\right)^2\right]\frac{\partial^2 u}{\partial x_2{}^2}\right\} = 0$$