Lecture 1

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Review: Finite - Dimensional / Inner - Product Spaces

Let $\mathcal{V}$ denote a finite - dimensional inner - product space:

(I) $\mathcal{V}$ is a vector space over $\mathbb{R}$ (or $\mathbb{C}$):

$\mathcal{V}$ = a collection (set) of vectors, typically denoted $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{a}, \underset{\sim}{c}$, etc., (i.e., lower-case symbols with a "tilde")

(i) vector addition: $\underset{\sim}{u}+\underset{\sim}{v} \in \mathcal{V} \quad \forall \underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{V}$

(ii) scalar multiplication: $\alpha \underset{\sim}{u} \in \mathcal{V} \ \forall \begin{cases} \alpha \in \mathbb{R} \\ \underset{\sim}{u} \in \mathcal{V} \end{cases}$

(iii) commutativity: $\underset{\sim}{u}+\underset{\sim}{v} = \underset{\sim}{v}+\underset{\sim}{u}$

(iv) associativity: $(\underset{\sim}{u}+\underset{\sim}{v})+\underset{\sim}{w} = \underset{\sim}{u}+(\underset{\sim}{v}+\underset{\sim}{w})$

(v) zero element: $\exists!\ \underset{\sim}{0} \in \mathcal{V}$ s.t.

$\underset{\sim}{0}+\underset{\sim}{u} = \underset{\sim}{u} \quad \forall\ \underset{\sim}{u} \in \mathcal{V}$.

(vi) negative element: for any $\underset{\sim}{u} \in \mathcal{V}$,

$\exists!\ -\underset{\sim}{u} \in \mathcal{V}$ s.t. $\underset{\sim}{u}+(-\underset{\sim}{u}) = \underset{\sim}{u}-\underset{\sim}{u} = \underset{\sim}{0}$.

(vii) $1\underset{\sim}{u} = \underset{\sim}{u} \quad \forall\ \underset{\sim}{u} \in \mathcal{V}$

(viii) $\alpha(\beta \underset{\sim}{u}) = (\alpha\beta)\underset{\sim}{u}$

(ix) $(\alpha+\beta)\underset{\sim}{u} = \alpha\underset{\sim}{u}+\beta\underset{\sim}{u}$

(x) $\alpha(\underset{\sim}{u}+\underset{\sim}{v}) = \alpha\underset{\sim}{u}+\alpha\underset{\sim}{v}$.

(2) V is equipped with an <u>inner-product</u>, denoted $\underset{\sim}{u}\cdot\underset{\sim}{v}$ (or sometimes we will also use $\langle \underset{\sim}{u}, \underset{\sim}{v} \rangle$), satisfying (for scalar field $\mathbb{R}$)

(i) $\underset{\sim}{u}\cdot\underset{\sim}{v} = \underset{\sim}{v}\cdot\underset{\sim}{u}$

(ii) $\underset{\sim}{u}\cdot(\underset{\sim}{v}+\underset{\sim}{w}) = \underset{\sim}{u}\cdot\underset{\sim}{v} + \underset{\sim}{u}\cdot\underset{\sim}{w}$

(iii) $(\alpha\underset{\sim}{u})\cdot\underset{\sim}{v} = \alpha(\underset{\sim}{u}\cdot\underset{\sim}{v})$

(iv) $\underset{\sim}{u}\cdot\underset{\sim}{u} > 0 \quad \forall\, \underset{\sim}{u} \neq \underset{\sim}{0}$.

Note: (iv) insures that $|\underset{\sim}{u}| \equiv (\underset{\sim}{u}\cdot\underset{\sim}{u})^{1/2}$ is a <u>norm</u> on V.

(3) $\exists$ a <u>basis</u> $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \dots, \underset{\sim}{e}_n\}$ for V, i.e

$$\underset{\sim}{v} = v^i \underset{\sim}{e}_i \left(\equiv \sum_{i=1}^{n} v^i \underset{\sim}{e}_i\right) \quad \forall\, \underset{\sim}{v} \in V$$

and $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \dots, \underset{\sim}{e}_n\}$ is linearly independent.

<u>Examples</u> (over $\mathbb{R}$)

(1) $\mathbb{R}^n$ $\quad \underset{\sim}{u} = (u_1, u_2, \dots, u_n)$

$$\underset{\sim}{u}\cdot\underset{\sim}{v} = u_1 v_1 + u_2 v_2 + \cdots u_n v_n = u_i v_i$$

(2) $\mathbb{M}^{m,n}$ $\quad m\times n$ real matrices

$$\underset{\sim}{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}$$

$$\underset{\sim}{A} \cdot \underset{\sim}{B} = a_{11} b_{11} + a_{12} b_{12} + \cdots + a_{mn} b_{mn}$$
$$= \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ij}$$

(3) $C^m([0,1])$ $\quad \underset{\sim}{f} \equiv f(x) \quad f : [0,1] \to \mathbb{R}$

m-times cont. differentiable

$$\underset{\sim}{f} \cdot \underset{\sim}{g} \equiv \int_0^1 f(x) g(x)\, dx$$

<u>Thm</u> Suppose that V is an inner-product space. Then

$$|\underset{\sim}{u} \cdot \underset{\sim}{v}| \leq |\underset{\sim}{u}||\underset{\sim}{v}| \quad \text{Cauchy-Schwarz ineq.}$$

$$|\underset{\sim}{u} + \underset{\sim}{v}| \leq |\underset{\sim}{u}| + |\underset{\sim}{v}| \quad \text{triangle ineq.}$$

<u>Pf</u> Assume $\underset{\sim}{u} \neq \underset{\sim}{0}$, $\underset{\sim}{v} \neq \underset{\sim}{0}$ (otherwise, if either one is zero, the result is trivial).

Observe $(\alpha \underset{\sim}{u} + \beta \underset{\sim}{v}) \cdot (\alpha \underset{\sim}{u} + \beta \underset{\sim}{v}) \geq 0$

$\forall\, \underset{\sim}{u}, \underset{\sim}{v} \in V$
$\alpha, \beta \in \mathbb{R}$

$$\Rightarrow \alpha^2 \underset{\sim}{u} \cdot \underset{\sim}{u} + 2\alpha\beta\, \underset{\sim}{u} \cdot \underset{\sim}{v} + \beta^2 \underset{\sim}{v} \cdot \underset{\sim}{v} \geq 0$$

In particular, choose $\alpha = |\underset{\sim}{v}|^2$, $\beta = -\underset{\sim}{u} \cdot \underset{\sim}{v}$

$$\Rightarrow |\underset{\sim}{v}|^4 |\underset{\sim}{u}|^2 - 2|\underset{\sim}{v}|^2 (\underset{\sim}{u} \cdot \underset{\sim}{v})^2 + (\underset{\sim}{u} \cdot \underset{\sim}{v})^2 |\underset{\sim}{v}|^2$$

(over

$$= |\underset{\sim}{v}|^2 \left[|\underset{\sim}{u}|^2 |\underset{\sim}{v}|^2 - (\underset{\sim}{u} \cdot \underset{\sim}{v})^2 \right] \geq 0$$

$$\underset{\sim}{v} \neq \underset{\sim}{0} \Rightarrow \quad \underset{\sim}{u} \cdot \underset{\sim}{v} \leq |\underset{\sim}{u}||\underset{\sim}{v}| \quad \checkmark$$

Next note

$$|\underset{\sim}{u} + \underset{\sim}{v}|^2 = (\underset{\sim}{u} + \underset{\sim}{v}) \cdot (\underset{\sim}{u} + \underset{\sim}{v})$$

$$= |\underset{\sim}{u}|^2 + 2(\underset{\sim}{u} \cdot \underset{\sim}{v}) + |\underset{\sim}{v}|^2$$

$$\underset{C-S}{\leq} |\underset{\sim}{u}|^2 + 2|\underset{\sim}{u}||\underset{\sim}{v}| + |\underset{\sim}{v}|^2$$

$$= (|\underset{\sim}{u}| + |\underset{\sim}{v}|)^2 . \quad \square$$

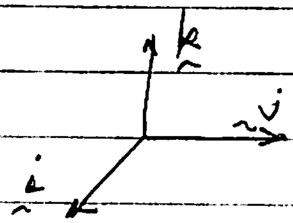
Reciprocal Basis

Consider first $\mathbb{R}^3$ (or $\mathbb{E}^3$) – Euclidean vector 3-space; Let

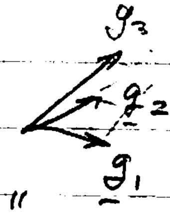
$\{\underset{\sim}{g}_1, \underset{\sim}{g}_2, \underset{\sim}{g}_3\}$ denote an arbitrary basis.

Let $\underset{\sim}{a} \times \underset{\sim}{b}$ denote the usual right-handed cross product, e.g.,

$$\underset{\sim}{i} \times \underset{\sim}{j} = \underset{\sim}{k}$$



Define
$$\boldsymbol{g}^1 \equiv \frac{\boldsymbol{g}_2 \times \boldsymbol{g}_3}{(\boldsymbol{g}_1 \times \boldsymbol{g}_2)\cdot \boldsymbol{g}_3}, \quad \text{"Vol"}$$



$$\boldsymbol{g}^2 \equiv \frac{\boldsymbol{g}_3 \times \boldsymbol{g}_1}{(\boldsymbol{g}_1 \times \boldsymbol{g}_2)\cdot \boldsymbol{g}_3},$$

$$\boldsymbol{g}^3 \equiv \frac{\boldsymbol{g}_1 \times \boldsymbol{g}_2}{(\boldsymbol{g}_1 \times \boldsymbol{g}_2)\cdot \boldsymbol{g}_3}.$$

<u>Claim</u> $\quad \boldsymbol{g}^i \cdot \boldsymbol{g}_j = \delta^i_j$

E.g.) $\quad \boldsymbol{g}^1 \cdot \boldsymbol{g}_1 = \dfrac{(\boldsymbol{g}_2 \times \boldsymbol{g}_3)\cdot \boldsymbol{g}_1}{(\boldsymbol{g}_1 \times \boldsymbol{g}_2)\cdot \boldsymbol{g}_3} = 1$

$$\boldsymbol{g}^1 \cdot \boldsymbol{g}_2 = \frac{\cancel{(\boldsymbol{g}_2 \times \boldsymbol{g}_3)\cdot \boldsymbol{g}_2}^{\,0}}{\text{Vol}} = 0 \checkmark$$

We call $\{\boldsymbol{g}^1, \boldsymbol{g}^2, \boldsymbol{g}^3\}$ the <u>reciprocal basis</u> or <u>dual basis</u>.

That it is indeed a basis, we will demonstrate shortly.

In general, suppose that $\mathcal{V}$ is a <u>real, n-dimensional inner-product space</u> with basis $\{\underset{\sim}{g}_1, \underset{\sim}{g}_2, \ldots, \underset{\sim}{g}_n\}$. Define

$$g_{ij} \equiv \underset{\sim}{g}_i \cdot \underset{\sim}{g}_j \qquad (\text{note } g_{ij} = g_{ji})$$

$$[G] = \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ & g_{22} & & \vdots \\ & \text{sym} & \ddots & \vdots \\ & & & g_{nn} \end{bmatrix} \quad n\times n, \text{ symmetric matrix}$$

<u>Proposition</u>. $[G]$ is invertible ($\det[G] \neq 0$)

<u>Proof</u>. Note
$$\begin{aligned} |\underset{\sim}{u}|^2 &= \underset{\sim}{u} \cdot \underset{\sim}{u} \\ &= (u^i \underset{\sim}{g}_i) \cdot (u^j \underset{\sim}{g}_j) \\ &= u^i u^j g_{ij} \quad > 0 \quad \forall\ \underset{\sim}{u} \neq \underset{\sim}{0} \end{aligned}$$

Assume that $[G]$ is not invertible. Then G has a zero eigenvalue, i.e., there is at least one n-tuple $(\phi^1, \phi^2, \ldots, \phi^n) \in \mathbb{R}^n$ such that

$$[G]\begin{pmatrix} \phi^1 \\ \phi^2 \\ \vdots \\ \phi^n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \text{ i.e.,}$$

or $\quad g_{ij}\phi^j = 0 \qquad i = 1, 2, \ldots, n$

$\Rightarrow \quad g_{ij}\phi^i\phi^j = 0$ — $*$ contradiction. □

Now

$$[G][G]^{-1} = [G]^{-1}[G] = [I] \leftarrow n\times n \text{ identity matrix}$$

Write $[G]^{-1} = \begin{bmatrix} g^{11} & g^{12} \cdots & g^{1n} \\ \vdots & & \vdots \\ g^{n1} & \cdots & g^{nn} \end{bmatrix}$

$\therefore \quad g_{ik}\, g^{kj} = \delta_i^j \qquad g^{ik} g_{kj} = \delta_j^i$

Define: $\boxed{\underset{\sim}{g}^i = g^{ij}\underset{\sim}{g}_j}$ $\quad i = 1, 2, \ldots, n,$

which are called the <u>reciprocal</u> or <u>dual</u> <u>basis vectors</u>.

<u>Note</u> $\quad \underset{\sim}{g}_i \cdot \underset{\sim}{g}^j = \underset{\sim}{g}_i \cdot g^{jk}\underset{\sim}{g}_k$

$$= g^{jk} g_{ki} = \delta_i^j$$

($\{\underset{\sim}{g}_1, \underset{\sim}{g}_2, \ldots, \underset{\sim}{g}_n\}$ $\{\underset{\sim}{g}^1, \underset{\sim}{g}^2, \ldots, \underset{\sim}{g}^n\}$ biorthonormal pair of bases)

<u>Thm</u>. $\{\underset{\sim}{g}^1, \underset{\sim}{g}^2, \ldots, \underset{\sim}{g}^n\}$ is also a basis for $\mathcal{V}$.

Pf. For any $\underset{\sim}{v} \in \mathcal{V}$, we have

$$\underset{\sim}{v} = v^i \underset{\sim}{g}_i$$

But $$\underset{\sim}{g}^i = g^{ij} \underset{\sim}{g}_j$$

$$\therefore \boxed{g_{ki} \underset{\sim}{g}^i = g_{ki} g^{ij} \underset{\sim}{g}_j = \delta^j_k \underset{\sim}{g}_j = \underset{\sim}{g}_k}$$

$$\hookrightarrow \quad \underset{\sim}{v} = \underbrace{(v^i g_{ij})}\underset{\sim}{g}^j = v_j \underset{\sim}{g}^j$$

$\Rightarrow \{\underset{\sim}{g}^1, \underset{\sim}{g}^2, \dots, \underset{\sim}{g}^n\}$ is a spanning set

Also $$c_1 \underset{\sim}{g}^1 + c_2 \underset{\sim}{g}^2 + \dots + c_n \underset{\sim}{g}^n = \underset{\sim}{0}$$

or $$c_i \underset{\sim}{g}^i = 0$$

$$\Rightarrow (c_i g^{ij}) \underset{\sim}{g}_j = \underset{\sim}{0}$$

But $\{\underset{\sim}{g}_1, \underset{\sim}{g}_2, \dots, \underset{\sim}{g}_n\}$ is a basis (by assumption) $\Rightarrow$

$$c_i g^{ij} = 0 \qquad G^T \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = 0$$

$[G]$ invertible $\Leftrightarrow$ $\det[G] \neq 0$ $\quad \det[G]^T = \det[G] \neq 0$

$$\Rightarrow \quad c_1 = c_2 = \cdots = c_n = 0 \; . \; \square$$

Basic properties:

$$\boxed{\underset{\sim}{g}^i \cdot \underset{\sim}{g}^j} = g^{ik}\underset{\sim}{e}_k \cdot g^{jl}\underset{\sim}{e}_l = g^{ik} g^{jl} g_{kl} \overset{(sym)}{=} g^{ik} g^{jl} g_{lk} = g^{ik}\delta_j^k = g^{ij}$$

$$\Rightarrow \quad \boxed{g^{ij} = g^{ji}} \qquad (G^{-1} \text{ is also a sym. matrix})$$

For any $\underset{\sim}{v} \in \mathcal{V}$, we have the unique representations

$$\underset{\sim}{v} = v_i \underset{\sim}{g}^i = v^i \underset{\sim}{g}_i \, .$$

We call the

v_i – covariant components of $\underset{\sim}{v}$ (relative to the given biorthonormal bases) ("co" goes below)

v^i – contra-variant components of $\underset{\sim}{v}$ (relative to the biorthonormal pair)

$$\longrightarrow \quad v_i \underset{\sim}{g}^i = v^j \overbrace{g_{ji} \underset{\sim}{g}^i}^{\underset{\sim}{g}_j} \overset{(sym)}{=} v^j g_{ij} \underset{\sim}{g}^i$$

$$\Rightarrow \quad \boxed{v_i = g_{ij} v^j} \qquad \text{"lowering an index"}$$

$$g^{ij}$$

$$V^i \underset{\sim}{g}_i = V_j \underset{\sim}{g}^j = V_j\, g^{ji} \underset{\sim}{g}_i$$

$$\Rightarrow \boxed{V^i = g^{ij} V_j} \quad \text{"raising of an index"}$$

Lecture 2

Also

$$\boxed{\underset{\sim}{g}_i \cdot \underset{\sim}{V} = \underset{\sim}{g}_i \cdot V_j \underset{\sim}{g}^j = V_j\, \delta^j_i = V_i}$$

$$\boxed{\underset{\sim}{g}^i \cdot \underset{\sim}{V} = \underset{\sim}{g}^i \cdot V^j \underset{\sim}{g}_j = V^j \delta^i_j = V^i}$$

Thm Given $\{\underset{\sim}{g}_1, \underset{\sim}{g}_2, \dots, \underset{\sim}{g}_n\}$ (a basis for $\mathcal{V}$), then the reciprocal basis $\{\underset{\sim}{g}^1, \underset{\sim}{g}^2, \dots, \underset{\sim}{g}^n\}$ (as constructed on p. 7) is <u>unique</u>.

Pf We argue by contradiction – suppose that $\{\underset{\sim}{d}^1, \underset{\sim}{d}^2, \dots, \underset{\sim}{d}^n\}$ is another reciprocal basis relative to $\{\underset{\sim}{g}_1, \underset{\sim}{g}_2, \dots, \underset{\sim}{g}_n\}$. Then

$$\underset{\sim}{g}^i \cdot \underset{\sim}{g}_j = \delta^i_j \quad \text{and} \quad \underset{\sim}{d}^i \cdot \underset{\sim}{g}_j = \delta^i_j .$$

Subtracting these two equations yields

$$(*) \quad (\underset{\sim}{g}^i - \underset{\sim}{d}^i) \cdot \underset{\sim}{g}_j = 0$$

Since $\{\underset{\sim}{g}_1, \underset{\sim}{g}_2, \dots, \underset{\sim}{g}_n\}$ is a basis, $(*)$ is equivalent to