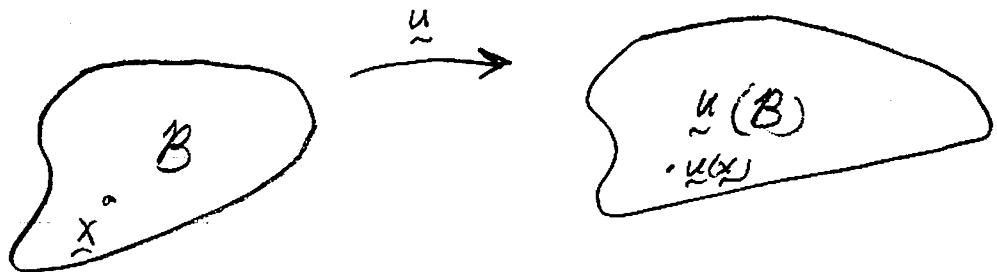
Lecture 11

(Examples cont.)

5. Nonlinear Elasticity



$\mathcal{B} \subset \mathbb{E}^3$ open

$\nabla \underset{\sim}{u}(\underset{\sim}{x})$ deformation gradient

$W(\underset{\sim}{A})$ stored energy function

Minimize total potential energy:

$$E = \iiint_{\mathcal{B}} \left[W(\nabla \underset{\sim}{u}) - \hat{\underset{\sim}{b}} \cdot \underset{\sim}{u} \right] dV - \iint_{\partial \mathcal{B}_1} \hat{\underset{\sim}{T}} \cdot \underset{\sim}{u} \, dS$$

body forces

surface tractions

Subj. to $\underset{\sim}{u}|_{\partial \mathcal{B}_2} = \hat{\underset{\sim}{u}}$.

In each of these cases, we are hoping to minimize <u>functionals</u>, i.e., <u>real-valued functions</u> defined on <u>classes of functions</u>.

Simplest Case

$$I(y) \equiv \int_a^b f(x, y, y')\, dx,$$

subject to geometric end conditions:

$$y(a) = y_a, \quad y(b) = y_b, \quad y_a, y_b \in \mathbb{R}.$$

(geometric boundary conditions)

Here $f : \overset{(\text{open})}{\mathcal{D}} \subset \mathbb{R}^3 \to \mathbb{R}$ is of class C^1.

Let $\mathbb{X}$ denote some class of functions appropriate for rendering $I(y)$ well-defined, e.g., $\mathbb{X} = C^1([a,b])$. We assume ~~that~~ $\mathbb{X}$ is a normed linear space (a vector space equipped with a norm $|\cdot|_{\mathbb{X}}$).

$\boxed{\text{Ex}}$ $\mathbb{X} = C^1([a,b])$ ($\equiv$ all continuously differentiable functions on $[a,b]$)

$$|u|_{\mathbb{X}} = \max_{a \le x \le b} |u(x)| + \max_{a \le x \le b} |u'(x)|.$$

Then $\mathbb{X}$ is linear, i.e., $\alpha u + \beta v \in \mathbb{X}$ $\forall\, u, v \in \mathbb{X}$, $\alpha, \beta \in \mathbb{R}$, and $|\cdot|_{\mathbb{X}}$ is a norm.

Now $I : \tilde{\mathbb{X}} \equiv \{\mathbb{X} : y(a) = y_a, y(b) = y_b\} \to \mathbb{R}$ is a functional. ($\tilde{\mathbb{X}}$ is not a linear space)

Defn We say that $I(\cdot)$ has a local minimum (maximum) at $y_0(x) \in \tilde{\mathbb{X}}$ if

$$(*) \quad I(y_0) \le (\ge)\, I(y) \quad \forall\ |y-y_0|_X < \varepsilon .$$

If $(*)$ holds $\forall\ y \in \tilde{X}$, then $I(\cdot)$ has a global minimum (maximum) at y_0.

<u>Remark</u> This set-up is tractable only in the case when X is <u>complete</u>, i.e., if $\{y_n\} \subset X$ converges, i.e., if

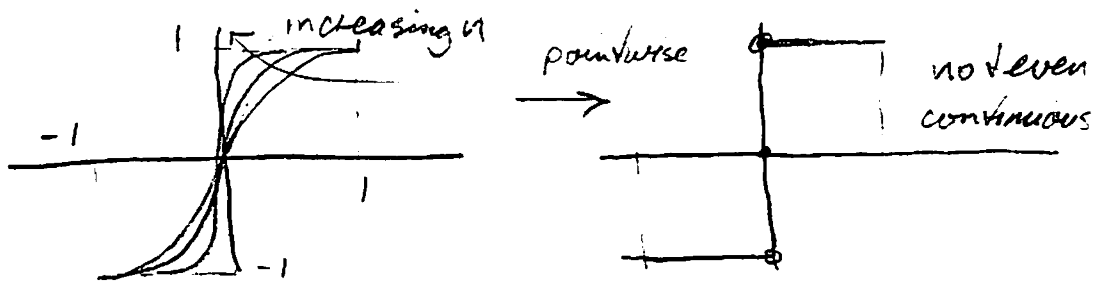
$$|y_n - y_*|_X \to 0 \quad n \to \infty,$$

then $y_* \in X$. In this case X is called a <u>Banach space</u>. For example, $C'([a,b])$ with $|\cdot|_X$ as given a p. 99 is indeed a Banach space. But $C'([a,b])$ with the norm

$$|y|_X^2 \equiv \int_a^b (y(x))^2\,dx \quad \text{is not.}$$

Indeed, let $a=-1$, $b=1$ and define the sequence of functions

$$y_n(x) = \frac{\tan^{-1}(nx)}{\tan^{-1}(n)} \qquad n = 1, 2, \ldots$$

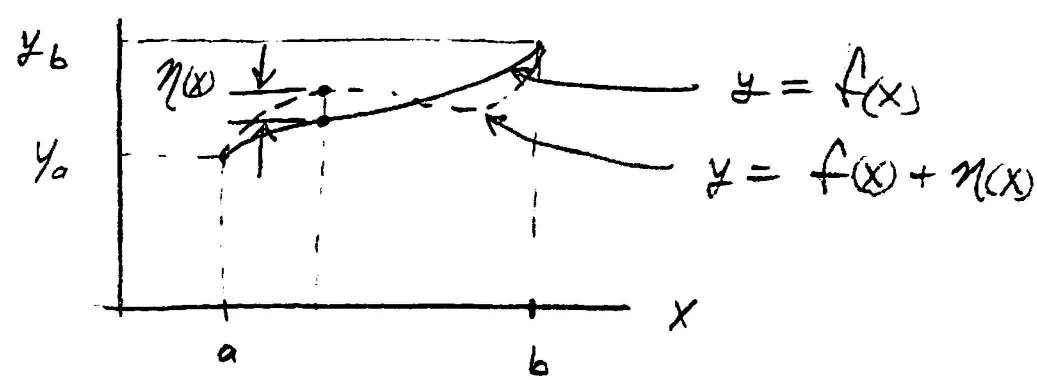


Returning to (*) p. 99, notice that, for $y, y_0 \in \tilde{X}$, then $y - y_0$ belongs to

$$X_0 = \{ X : \underbrace{y(a) = y(b) = 0}_{\text{homogeneous b.c.'s}} \}.$$

<u>Defn</u> We call X_0 the space of <u>admissible variations</u>.

In essence, if $y = f(x) \in \tilde{X}$ is a candidate minimizer (maximizer) of $I(y)$, ~~then~~ $f(x) + \eta(x) \in \tilde{X}$ for all $\eta \in X$.



We now assume ~~that~~ (*) holds to derive a necessary condition for a minimum (maximum), directly imitating what we did on p. 89, i.e., if (*) holds, ~~then~~

$$g(\alpha) \equiv I(y_0 + \alpha\eta) \quad \left(g : \mathbb{R} \to \mathbb{R} \right)$$

has a local minimum (maximum) at $\alpha = 0$. Thus, $g'(0) = 0 \Rightarrow$

$$(**) \quad \delta I = \frac{d}{d\alpha} I(y_0 + \alpha\eta)\Big|_{\alpha=0} = 0$$

$\forall$ admissible variations $\eta \in X_0$,

is necessary for $I(\cdot)$ to have a local extremum at $y_0 \in \tilde{X}$.

The left-hand side of $(**)$ is called the 1st variation and $(**)$ itself is called the 1st variation condition – its nothing more than a generalization of the 1st derivative test from calculus!

Let's proceed with the calculation in $(**)$, assuming the structure for X in the example on p. 98:

$$\frac{d}{d\alpha} I(y+\alpha\eta)\Big|_{\alpha=0} = \frac{d}{d\alpha}\int_a^b f(x, y+\alpha\eta, y'+\alpha\eta')\,dx\Big|_{\alpha=0}$$

$$= \int_a^b \Big[\frac{\partial f}{\partial y}(x,y,y')\eta(x) + \underbrace{\frac{\partial f}{\partial y'}(x,y,y')\eta'(x)}_{\text{integrate by parts}}\Big]dx$$

$$u = \frac{\partial f}{\partial y'}(x,y,y') \qquad \frac{dv}{dx} = \eta'$$

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \qquad v = \eta$$

$(\eta(a) = \eta(b) = 0)$

$$\therefore \quad \delta I = \int_a^b \frac{\partial f}{\partial y}\,\eta\,dx + \cancel{\frac{\partial f}{\partial y'}(x,y,y')\,\eta(x)\Big|_a^b}^{\,0}$$

$$- \int_a^b \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta\,dx = 0 \qquad \forall \eta \in X_0$$

$$\therefore \quad \delta I = \int_a^b \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\eta\,dx = 0$$

In order to proceed, we require

<u>Localization Lemma.</u> If $g : [a,b] \to \mathbb{R}$ is continuous and

$$\int_a^b g(x)\,\eta(x)\,dx = 0 \quad \forall \eta \in X_0,$$

then $F(x) \equiv 0$ on $[a,b]$.

<u>Pf.</u> Suppose $F(x_0) > 0$ (say) for $a \le x_0 \le b$. Then by continuity, $F(x) > 0$ on some interval $(x_1, x_2) \subset [a,b]$. Choose

$$\eta(x) \begin{cases} > 0 & x_1 < x < x_2 \\ = 0 & \text{otherwise} \end{cases}, \text{ with } \eta \in X_0,$$

e.g., $\eta(x) = (x-x_1)^2(x-x_2)^2 \quad x_1 < x < x_2$.

Then

contradiction

$$\int_a^b g(x)\,\eta(x)\,dx = \int_{x_1}^{x_2} g(x)\,\eta(x)\,dx > 0. \quad \square$$

Application of this result to the 1st var. condition (p. 102) yields the

Euler-Lagrange equation:

$$\boxed{\begin{aligned} &\frac{\partial f}{\partial y}(x, y, y') - \frac{d}{dx}\frac{\partial f}{\partial y'}(x, y, y') = 0, \\ &\text{subj. to} \quad y(a) = y_a, \; y(b) = y_b. \end{aligned}}$$

Ex) Euclidean geodesic problem (p. 94):

$$I = \int_a^b \underbrace{\sqrt{1+(y')^2}}_{f}\, dx$$

$$\frac{\partial f}{\partial y} = 0 \qquad \frac{\partial f}{\partial y'} = \frac{1}{\cancel{2}}\frac{\cancel{2}y'}{\sqrt{1+(y')^2}}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) = 0$$

$$\Rightarrow \frac{y'}{\sqrt{1+(y')^2}} = C$$

$$\text{or} \quad (y')^2 = C^2\left(1+(y')^2\right)$$

$$\Rightarrow (y')^2 = K^2 \quad (K \text{ const})$$

$$\Rightarrow y' = m$$

$$\text{or } y = mx + c \qquad \underline{\text{Straight line}}$$

<u>Remark</u>. A solution of the Euler-Lagrange equation satisfying the geometric end conditions is called a <u>critical point</u> of I.

Now our derivation of the Euler-Lagrange equation, assuming $y \in C^1([a,b])$, suffers from the weakness that we have presumed that $\frac{\partial f}{\partial y'}(x,y,y')$ is continuously differentiable, which need not be true. In fact, we don't need that assumption:

<u>Lemma 1</u> If $h(x)$ is continuous on $[a,b]$ and if

$$\int_a^b h(x)\,\eta'(x)\,dx = 0 \qquad \forall\, \eta \in I_0,$$

then $h(x) \equiv C$ (const.) on $[a,b]$.

<u>Pf</u> Let C denote the unique constant such ~~that~~

$$\int_a^b h(x)\,dx = C(b-a)$$

$$\Rightarrow \quad \int_a^b \big(h(x) - C\big)\,dx = 0$$

Define $\quad \eta(x) \equiv \int_a^x \big(h(\tau) - C\big)\,d\tau$.

Then $\eta'(x) = h(x) - c$ (FTC), and

$$\int_a^b (h(x)-c)^2 dx = \int_a^b (h(x)-c)\eta'(x)\,dx$$

$$= \int_a^b h(x)\,\eta'(x)\,dx - c\,\cancel{\eta}\Big|_a^b{}^{\,0} = 0$$

$$\Leftrightarrow \quad h(x) \equiv c. \quad \square$$

<u>Lemma 2</u> If $g(x)$ and $h(x)$ are each continuous on $[a,b]$, and if

$$\int_a^b [g(x)\,\eta(x) + h(x)\,\eta'(x)]\,dx = 0 \quad \forall \eta \in \mathcal{I}_0,$$

then $h(x)$ is differentiable with $h'(x) \equiv g(x)$.

<u>Pf</u> <u>Consider</u> $\displaystyle\int_a^b g(x)\,\eta(x)\,dx$, and

integrate by parts:
$$u = \eta \qquad \frac{dv}{dx} = g(x)$$
$$\frac{du}{dx} = \eta' \qquad v = \int_a^x g(\tau)\,d\tau \equiv \ell(x)$$

$$= \cancel{\ell(x)\,\eta(x)}\Big|_a^b{}^{\,0} - \int_a^b \ell(x)\,\eta'(x)\,dx$$

$$\therefore \int_a^b [h(x) - \ell(x)]\,\eta'(x)\,dx = 0 \quad \forall \eta \in \mathcal{I}_0$$

$$\overset{\text{Lemma 1}}{\Longrightarrow} \quad h(x) = \int_a^x g(\tau)\,d\tau + C. \quad \square$$

Returning to δI on p. 101, we have

$$(**) \quad \delta I = \int_a^b \left[\frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta'\right] dx = 0 \quad \forall\, \eta \in X_0.$$

Thus, by Lemma 2, we rigorously deduce the Euler-Lagrange eq. on p. 103, i.e., $\frac{\partial f}{\partial y'}(x, y, y')$ is differentiable with

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}(x, y, y')\right) = \frac{\partial f}{\partial y}(x, y, y') .$$

<u>Remark</u> If we relax the condition ~~that~~ $y \in C^1([a,b])$, the 1st variation condition $(**)$ above is called the weak form of the Euler-Lagrange equation. E.g., y' piecewise continuous is often more appropriate - as we shall see later.

<u>Some "Tricks of the Trade"</u>

1. If f is independent of y, then E-L eq'n reduces to

$$\frac{\partial f}{\partial y'}(x, y') = \text{const}.$$

2. If f is independent of y', then

$$\frac{\partial f}{\partial y}(x, y) = 0 \quad \text{alg. eq'n for } y \text{ as funct of } x$$

(curve in the x-y plane – generally won't satisfy the geom. end conditions

3. If f is independent of x, ~~then~~

$$\frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial y}y' + \cancel{\frac{\partial f}{\partial y'}y''}$$

$$\cancel{-y''\frac{\partial f}{\partial y'}} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)y'$$

$$= y'\underbrace{\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]}_{0}$$

$$\Rightarrow \quad f - y'\frac{\partial f}{\partial y'} = \text{Const}$$

<u>Ex</u>

$$I = \int_a^b [\cos y - x\sin(y)\,y']\,dx$$

$$\frac{\partial f}{\partial y} = -\sin y - x\cos(y)\,y'$$

$$\frac{\partial f}{\partial y'} = -x\sin(y)$$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = -\sin y - x\cos(y)\,y' - \left(-\sin y - x\cos(y)\,y'\right) \equiv 0$$

What happened? Apparently any function $y(x)$ will work.

More generally if

$$I = \int_a^b \left(M(x,y) + N(x,y)\,y'\right)dx$$

and

$$\frac{\partial M}{\partial y} = -\frac{\partial N}{\partial x}$$

then I is path-independent, i.e., I is const. on all smooth paths!

Ex) Find the curve $y(x)$ that connects $(0,0)$ to $(1,1)$ and that has the least moment of inertia about the x-axis:

$$I(y) = \int_0^1 y\sqrt{1+(y')^2}\,dx$$

$$y(0) = 0, \quad y(1) = 1$$