Similarly, the minimum principle asserts

$$w = u_1(x,t) - u_2(x,t) \geq \min_{0\leq x\leq l} (\hat{\phi}_1 - \hat{\phi}_2)$$

$$\geq -\max_{0\leq x\leq l} |\hat{\phi}_1 - \hat{\phi}_2|$$

$$\Rightarrow \quad \max_{\substack{0\leq x\leq l \\ t>0}} |u_1 - u_2| \leq \max_{0\leq x\leq l} |\hat{\phi}_1 - \hat{\phi}_2|$$

Well-posed in the uniform sense.

Claim The heat equation is not well-posed for $t<0$ (backward heat equation). To see this, consider the sequence

$$u_n(x,t) = \frac{1}{n}\sin nx \, e^{-n^2\alpha^2 t} \qquad n = 1, 2, \ldots$$

For each $n \in \mathbb{Z}^+$, $u_n(x,t)$ satisfies the heat equation for the I.C.

$$u_n(x,0) = \frac{1}{n}\sin nx$$

In fact it is the unique solution of

$$\left.\begin{aligned} &u_t = \alpha^2 u_{xx} \\ &u(0,t) = u(l,t) = 0 \\ &u(x,0) = \tfrac{1}{n}\sin nx \end{aligned}\right\} \begin{aligned} &0<x<l \\ &-\infty<t<\infty \end{aligned}$$

Note ~~the~~ $u(x,0) = \frac{1}{n} \sin nx \to 0$ uniformly as $n \to \infty$.

But for any $t < 0$, say, $t = -1$, we have

$$u(x,-1) = \frac{1}{n} e^{n^2\alpha^2} \sin nx \to \pm\infty$$

uniformly as $n \to \infty$

Conclusion: For zero initial data,

$$\left.\begin{aligned} &u_t = \alpha^2 u_{xx} \\ &u(0,t) = u(l,t) = 0 \\ &u(x,0) = 0 \end{aligned}\right\} \begin{aligned} &0 < x < l \\ &-\infty < t < \infty \end{aligned},$$

we have the unique solution $u \equiv 0$.

Now choose n sufficiently large $\Rightarrow$

$u(x,0) = \frac{1}{n} \sin nx$ is as small as we like (uniformly).

Yet $u(x,t)$ is arbitrarily large for each $t < 0$.

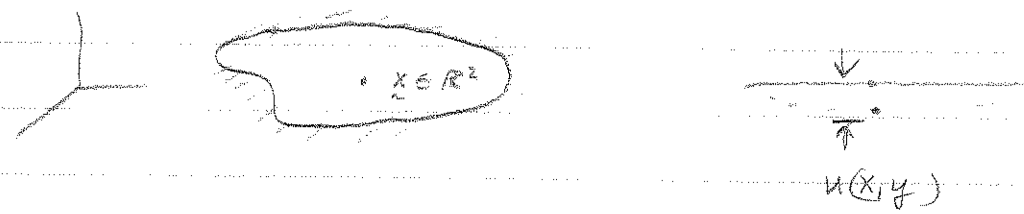
$\Rightarrow$ ill-posed for $t < 0$.

Variational Characterization of Eigenvalues

Consider the 2-dim. wave equation

$$(*) \quad \left.\begin{array}{l} u_{tt} = c^2 \Delta u \\ u|_{\partial\Omega} = 0 \end{array}\right\} \begin{array}{l} \text{on } \Omega \subset \mathbb{R}^2 \\ t > 0 \end{array}$$

where Ω is a bounded smooth domain. This model, e.g., the small transverse motions of a membrane in the shape of Ω (a drum).



In vibration theory one looks for

$$u(\underline{x}, y, t) = u(x, y)\, e^{i\omega t} \quad (\text{sep. variable})$$

Subst. into $(*)$ yields the Helmholst equation

$$\Delta u + \lambda u = 0 \quad \text{in } \Omega$$
$$u|_{\partial\Omega} = 0,$$

where $\lambda \equiv \dfrac{\omega^2}{c^2}$

This is an eigenvalue problem for the operator $-\Delta(\ )$ s.t. Dirichlet b.c.'s

$$(*)\quad \begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

We seek nontrivial solutions $u \not\equiv 0$ (observe that $u \equiv 0$ is always a solution) called eigenfunctions corresponding to eigenvalues λ.

Definition $\quad Q(w) \equiv \dfrac{\int_\Omega |\nabla w|^2 \, d\underline{x}}{\int_\Omega w^2 \, d\underline{x}}$

is called the Rayleigh quotient

Consider the minimum problem

$$m \equiv \min \left\{ Q(w) : w \in C^2(\overline{\Omega}),\ w \not\equiv 0 \text{ in } \Omega,\ w|_{\partial\Omega} = 0 \right\}$$

This is equivalent to the constrained minimization problem

$$\min I(w) = \int_\Omega |\underline{\nabla} w|^2 \, d\underline{x}$$

subject to $\int_\Omega w^2 \, d\underline{x} = 1$.

Claim $m = \lambda_1$, where λ_1 is the smallest eigenvalue of $(*)$.

Pf Let $u(\underline{x})$ be the presumed solution of the min problem for $Q(w)$ (cf. p. 266); i.e.,

$$m = \frac{\int_\Omega |\nabla u|^2 d\underline{x}}{\int_\Omega u^2 d\underline{x}} \leq \frac{\int_\Omega |\nabla w|^2 d\underline{x}}{\int_\Omega w^2 d\underline{x}} \equiv Q(w)$$

for all trial functions $w \in C^2(\bar{\Omega})$, $w \not\equiv 0$ in Ω, $w|_{\partial\Omega} = 0$.

Let's take the 1st variation of Q:

$$\delta Q = \frac{d}{d\varepsilon} Q(u + \varepsilon\eta)\Big|_{\varepsilon=0} = 0 \quad \forall\ \eta \text{ trial functions}$$

$$= \frac{d}{d\varepsilon} \frac{\int_\Omega (\nabla u + \varepsilon\nabla\eta)\cdot(\nabla u + \varepsilon\nabla\eta)\, d\underline{x}}{\int_\Omega (u+\varepsilon\eta)^2 d\underline{x}}\Bigg|_{\varepsilon=0}$$

$$= \frac{2\int_\Omega \nabla u \cdot \nabla\eta\, d\underline{x}}{\int_\Omega u^2 d\underline{x}} - \frac{2\int_\Omega |\nabla u|^2 d\underline{x} \int_\Omega u\eta\, d\underline{x}}{\left(\int_\Omega u^2 d\underline{x}\right)^2}$$

But $m = \dfrac{\int_\Omega |\nabla u|^2 d\underline{x}}{\int_\Omega u^2 d\underline{x}} \Rightarrow$

$$\frac{\delta Q}{2} = \frac{\int_\Omega \nabla u \cdot \nabla \eta \, d\underset{\sim}{x}}{\int_\Omega u^2 \, d\underset{\sim}{x}} - m \frac{\int_\Omega u \eta \, d\underset{\sim}{x}}{\int_\Omega u^2 \, d\underset{\sim}{x}} = 0 \quad \forall \eta$$

Finally, integrate by parts:

$$\underset{\sim}{\nabla} \cdot (\eta \underset{\sim}{\nabla} u) = \eta \, \Delta u + \underset{\sim}{\nabla} u \cdot \underset{\sim}{\nabla} \eta \quad \text{(HW 8, prob 1)}$$

$$\int_\Omega \nabla u \cdot \nabla \eta \, d\underset{\sim}{x} = \int_\Omega \underset{\sim}{\nabla} \cdot (\eta \nabla u) d\underset{\sim}{x} - \int_\Omega \eta \, \Delta u \, d\underset{\sim}{x}$$

$$\overset{\text{div thm}}{=} \int_{\partial\Omega} \nabla u \cdot \underset{\sim}{n} \, \underset{0}{\cancel{\eta}} \, ds - \int_\Omega \Delta u \, \eta \, d\underset{\sim}{x}$$

$$\Rightarrow \frac{\delta Q}{2} = - \frac{\int_\Omega (\Delta u + mu) \eta \, d\underset{\sim}{x}}{\int_\Omega u^2 \, d\underset{\sim}{x}} = 0 \quad \forall \eta$$

$$\Rightarrow \quad \Delta u + m u = 0 \quad \text{in } \Omega$$

$$u|_{\partial\Omega} = 0 \quad \text{geo. b.c.}$$

$(u \not\equiv 0$ admissible$)$

$\Rightarrow$ m is an eigenvalue and $u(\underset{\sim}{x})$ the corresponding eigenfunction.

Why is $m \equiv \lambda_1$ the smallest eigenvalue?
Let λ_*, $v_*(\underset{\sim}{x})$ be any other eigenpair.

Then $$m \le \frac{\int_\Omega |\nabla v_*|^2 \, d\underline{x}}{\int_\Omega v_*^2 \, d\underline{x}}$$

But (again) $\nabla \cdot (v_* \nabla v_*) = |\nabla v_*|^2 + v_* \Delta v_*$

and $\int_\Omega |\nabla v_*|^2 d\underline{x}$

$$= \int_\Omega \left(\nabla \cdot (v_* \nabla v_*) - v_* \Delta v_*\right) d\underline{x}$$

$$\overset{\text{div thm}}{=} -\int_\Omega v_* \Delta v_* \, d\underline{x} + \int_{\partial\Omega} \cancel{v_*} \nabla v_* \cdot \underline{n} \, ds$$

$$\therefore \quad m \le \frac{-\int_\Omega \Delta v_* \, v_* \, d\underline{x}}{\int_\Omega v_*^2 \, d\underline{x}} = \lambda_* \frac{\int_\Omega v_*^2 \, d\underline{x}}{\int_\Omega v_*^2 \, d\underline{x}} . \qquad \square$$

<u>Remark</u> The higher eigenvalues have a similar characterization, e.g.,

$$\lambda_2 = \min \left\{ Q(w) : w \in C^2(\bar{\Omega}), \, w|_{\partial\Omega} = 0, \, w \neq 0 \ \underline{\text{and}} \ \int_\Omega w \, u_1 \, d\underline{x} = 0 \right\}$$

It turns out ~~that~~ λ_1 is always simple — only one (positive) eigenfunction (given to within a constant). For the higher eigenvalues, we require w to be "$L^2 \perp$" to all eigenspaces of the lower eigenvalues.

Finally, we follow up on the claim at the bottom of p. 266:

$$\hat{I}(w,\lambda) = \int_\Omega \left[|\nabla w|^2 - \lambda w^2 \right] d\underline{x}$$

$$\delta I = \int_\Omega \left(\underline{\nabla} w \cdot \underline{\nabla} \eta - \lambda w \eta \right) d\underline{x}$$

$$\overset{\text{parts}}{\underset{\text{b.c.}}{=}} - \int_\Omega (\Delta w + \lambda w)\eta \, d\underline{x} = 0 \quad \forall \eta$$

$$\Rightarrow \text{E-L} \quad \Delta w + \lambda w = 0$$

$$w|_{\partial\Omega} = 0,$$

i.e., the Lagrange multiplier <u>is</u> the eigenvalue.