A class of universal relations in isotropic elasticity theory

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1. Introduction

Universal relations are equations that connect the components of stress and deformation tensors that hold for every material in a specified class.

One example is given for simple shear in lecture (Lecture notes page63-65) that for the given deformation $\mathbf{f}(\mathbf{X}) = (X_1 + kX_2)\mathbf{e}_1 + X_2\mathbf{e}_2 + X_3\mathbf{e}_3$, the following is independent of material properties for isotropic bodies:

$$T_{11} - T_{22} = kT_{12}$$

This is actually the first universal relation found and is by Rivlin.

In this presentation, some concepts are given first. Then a class of universal relations in isotropic elastic theory by Beatty [1] is investigated followed by one example. Finally, a more general scheme in finding linear universal relation by Bustamante and Ogden [2] is presented. In a linear universal relation, the components of the stress tensor appear linearly and they appear nonlinearly in a nonlinear universal relation.

2. Some concepts

Controllable deformation: A deformation that can be produced in a material by application of surface tractions alone.

Universal deformation: A controllable deformation that can be effected in every homogeneous, isotropic, hyper-elastic material is called a universal deformation.

Ericksen's problem: The problem of determine all such universal deformations for the two classes of compressible and incompressible, homogeneous and isotropic hyper-elastic material.

Ericksen [3] has shown that **the only deformation controllable in EVERY compressible homogeneous isotropic elastic materials are homogeneous.**
Shield [4] provided a more direct proof.
Proof: For isotropic materials:

$$W(\mathbf{F}) = \Phi(I_C, II_C, III_C) \tag{1}$$

For hyper-elastic materials:

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}} = \frac{\partial \Phi}{\partial \mathbf{F}} \tag{2}$$

Equilibrium holds:

$$\nabla \cdot \mathbf{S} = 0 \tag{3}$$

(3) gives us:

$$\nabla\bullet(\frac{\partial\Phi}{\partial I_c}\frac{\partial I_c}{\partial \mathbf{F}}+\frac{\partial\Phi}{\partial II_c}\frac{\partial II_c}{\partial \mathbf{F}}+\frac{\partial\Phi}{\partial III_c}\frac{\partial III_c}{\partial \mathbf{F}})=0$$

$$\begin{aligned}
\Rightarrow & \frac{\partial\Phi}{\partial I_c}(\nabla\bullet\frac{\partial I_c}{\partial \mathbf{F}})+\frac{\partial I_c}{\partial \mathbf{F}}(\nabla\frac{\partial\Phi}{\partial I_c})+\frac{\partial\Phi}{\partial II_c}(\nabla\bullet\frac{\partial II_c}{\partial \mathbf{F}})+\\
& \frac{\partial II_c}{\partial \mathbf{F}}(\nabla\frac{\partial\Phi}{\partial II_c})+\frac{\partial\Phi}{\partial III_c}(\nabla\bullet\frac{\partial III_c}{\partial \mathbf{F}})+\frac{\partial III_c}{\partial \mathbf{F}}(\nabla\frac{\partial\Phi}{\partial III_c})=0\\
\Rightarrow & \frac{\partial\Phi}{\partial I_c}(\nabla\bullet\frac{\partial I_c}{\partial \mathbf{F}})+\frac{\partial I_c}{\partial \mathbf{F}}(\frac{\partial^2\Phi}{\partial I_c\partial I_c}\nabla I_c+\frac{\partial^2\Phi}{\partial II_c\partial I_c}\nabla II_c+\frac{\partial^2\Phi}{\partial III_c\partial I_c}\nabla III_c)\\
& +\frac{\partial\Phi}{\partial II_c}(\nabla\bullet\frac{\partial II_c}{\partial \mathbf{F}})+\frac{\partial II_c}{\partial \mathbf{F}}(\frac{\partial^2\Phi}{\partial I_c\partial II_c}\nabla I_c+\frac{\partial^2\Phi}{\partial II_c\partial II_c}\nabla II_c+\frac{\partial^2\Phi}{\partial III_c\partial II_c}\nabla III_c)\\
& +\frac{\partial\Phi}{\partial III_c}(\nabla\bullet\frac{\partial III_c}{\partial \mathbf{F}})+\frac{\partial III_c}{\partial \mathbf{F}}(\frac{\partial^2\Phi}{\partial I_c\partial III_c}\nabla I_c+\frac{\partial^2\Phi}{\partial II_c\partial III_c}\nabla II_c+\frac{\partial^2\Phi}{\partial III_c\partial III_c}\nabla III_c)\\
& =0\\
\Rightarrow & \frac{\partial\Phi}{\partial I_c}(\nabla\bullet\frac{\partial I_c}{\partial \mathbf{F}})+\frac{\partial\Phi}{\partial II_c}(\nabla\bullet\frac{\partial II_c}{\partial \mathbf{F}})+\frac{\partial\Phi}{\partial III_c}(\nabla\bullet\frac{\partial III_c}{\partial \mathbf{F}})\\
& +\frac{\partial I_c}{\partial \mathbf{F}}[\nabla I_c]\frac{\partial^2\Phi}{\partial I_c\partial I_c}+\frac{\partial II_c}{\partial \mathbf{F}}[\nabla II_c]\frac{\partial^2\Phi}{\partial II_c\partial II_c}+\frac{\partial III_c}{\partial \mathbf{F}}[\nabla III_c]\frac{\partial^2\Phi}{\partial III_c\partial III_c}\\
& +(\frac{\partial I_c}{\partial \mathbf{F}}[\nabla II_c]+\frac{\partial II_c}{\partial \mathbf{F}}[\nabla I_c])\frac{\partial^2\Phi}{\partial I_c\partial II_c}\\
& +(\frac{\partial I_c}{\partial \mathbf{F}}[\nabla III_c]+\frac{\partial III_c}{\partial \mathbf{F}}[\nabla I_c])\frac{\partial^2\Phi}{\partial I_c\partial III_c}\\
& (\frac{\partial II_c}{\partial \mathbf{F}}[\nabla III_c]+\frac{\partial III_c}{\partial \mathbf{F}}[\nabla II_c])\frac{\partial^2\Phi}{\partial II_c\partial III_c}=0 \qquad (4)
\end{aligned}$$

In order that (4) holds for arbitrary choice of Φ, it is necessary and sufficient that the coefficients of each distinct derivatives of Φ vanish in (4).
Thus, we have:

$$\nabla\bullet\frac{\partial I_c}{\partial \mathbf{F}}=0 \qquad \text{(a)}$$

$$\nabla\bullet\frac{\partial II_c}{\partial \mathbf{F}}=0 \qquad \text{(b)}$$

$$\nabla\bullet\frac{\partial III_c}{\partial \mathbf{F}}=0 \qquad \text{(c)}$$

$$\frac{\partial I_c}{\partial \mathbf{F}}[\nabla I_c]=0 \qquad \text{(d)}$$

$$\frac{\partial II_c}{\partial \mathbf{F}}[\nabla II_c]=0 \qquad \text{(e)}$$

$$\frac{\partial III_C}{\partial \mathbf{F}}[\nabla III_C] = 0 \qquad \text{(f)}$$

$$\frac{\partial I_C}{\partial \mathbf{F}}[\nabla II_C] + \frac{\partial II_C}{\partial \mathbf{F}}[\nabla I_C] = 0 \qquad \text{(g)}$$

$$\frac{\partial I_C}{\partial \mathbf{F}}[\nabla III_C] + \frac{\partial III_C}{\partial \mathbf{F}}[\nabla I_C] = 0 \qquad \text{(h)}$$

$$\frac{\partial II_C}{\partial \mathbf{F}}[\nabla III_C] + \frac{\partial III_C}{\partial \mathbf{F}}[\nabla II_C] = 0 \qquad \text{(i)}$$

Since $I_C = tr(\mathbf{C}) = \mathbf{F}\bullet\mathbf{F}$, (a) indicates that

$$\nabla\bullet\mathbf{F} = 0 \quad \Rightarrow \quad \nabla\bullet(\nabla\mathbf{f}) = 0 \qquad \text{(j)}$$

Since $III_C = (\det \mathbf{F})^2$, (c) gives that

$$\nabla\bullet(2\det\mathbf{F}\frac{\partial \det\mathbf{F}}{\partial \mathbf{F}}) = 0$$

$$\Rightarrow \frac{\partial \det\mathbf{F}}{\partial \mathbf{F}}(\nabla \det\mathbf{F}) + \det\mathbf{F}(\nabla\bullet\frac{\partial \det\mathbf{F}}{\partial \mathbf{F}}) = 0$$

With Identities of Euler-Piola-Jacobi:

$$\frac{\partial \det\mathbf{F}}{\partial \mathbf{F}} = cof\mathbf{F} \equiv (\det\mathbf{F})\mathbf{F}^{-T} \quad \text{(refer to assignment 10)}$$

$$\nabla\bullet cof\mathbf{F} = 0 \quad \text{(refer to lecture notes page 90a)}$$

We have

$$cof\mathbf{F}(\nabla\det\mathbf{F}) = 0$$

$$\Rightarrow (\det\mathbf{F})\mathbf{F}^{-T}(\nabla\det\mathbf{F}) = 0$$

With the requirement $\det\mathbf{F} > 0$, $\mathbf{F}^{-T}$ is nonsigular,

$$\nabla\det\mathbf{F} = 0$$

$$\Rightarrow \det\mathbf{F} \text{ is constant} \qquad \text{(k)}$$

Finally (d) gives that

$$2\mathbf{F}(\nabla I_C) = 0$$

F is nonsigular,

$$\nabla I_C = 0$$

$$\Rightarrow \nabla\bullet(\nabla I_C) = 0$$

Now work in components:

$$\frac{\partial}{\partial X_k}\frac{\partial F_{ij}F_{ij}}{\partial X_k} = 0$$

$$\Rightarrow \frac{\partial}{\partial X_k}(F_{ij}\frac{\partial F_{ij}}{\partial X_k}) = 0$$

$$\Rightarrow \frac{\partial F_{ij}}{\partial X_k}\frac{\partial F_{ij}}{\partial X_k} + F_{ij}\frac{\partial^2 F_{ij}}{\partial X_k^2} = 0$$

$$\Rightarrow \frac{\partial^2 f_i}{\partial X_j \partial X_k}\frac{\partial^2 f_i}{\partial X_j \partial X_k}+\frac{\partial f_i}{\partial X_j}\frac{\partial^3 f_i}{\partial X_j \partial X_k^2}=0$$

The second term above vanishes due to (j).

$$\Rightarrow \frac{\partial^2 f_i}{\partial X_j \partial X_k}=0$$

f is linear in **X**, i.e. **F** is constant, thus the deformation is homogeneous.
All the others ((b),(e)-(i)) are then satisfied automatically.
Ericksen also has a discussion on deformations possible in every isotropic, incompressible, perfectly elastic body. But we do not investigate this in details here.

3. A class of universal relations for isotropic material

The constitutive relation for an isotropic material is given by:

$$\mathbf{T}=\tilde{\psi}_0(I_B,II_B,III_B)\mathbf{I}+\tilde{\psi}_1(I_B,II_B,III_B)\mathbf{B}+\tilde{\psi}_{-1}(I_B,II_B,III_B)\mathbf{B}^{-1} \tag{5}$$

If the material is incompressible, then the constitutive relation could be expressed as:

$$\mathbf{T}=-p\mathbf{I}+\tilde{\phi}_0(I_B,II_B)\mathbf{B}+\tilde{\phi}_1(I_B,II_B)\mathbf{B}^2 \tag{6}$$

From (5) and (6) we observe that

$$\mathbf{TB}=\mathbf{BT} \tag{*}$$

(*) is necessary and sufficient for the coincidence of principal direction of symmetric tensors **T** and **B**. It is obvious that the principal directions of **T** coincide with those of **B**.

Claim: The principal directions of **B** coincide with those of **T**, and if two or three of the principal values of **T** are equal, **B** also has corresponding equal principal values, provided that the empirical inequalities hold:

$$\tilde{\psi}_1>0 \text{ and } \tilde{\psi}_{-1}\le 0 \tag{7}$$

Proof: (Batra [5])
For compressible materials, when **T** is written in terms of its principal values in the principal coordinate:

$$\mathbf{T}=\sum_{i=1,2,3} T_{ii}\mathbf{E}_i\otimes\mathbf{E}_i \tag{8}$$

(*) requires that

$$B_{12}(T_1-T_2)=B_{23}(T_2-T_3)=B_{13}(T_1-T_3)=0 \tag{9}$$

If $T_1\neq T_2\neq T_3$,

$B_{12}=B_{23}=B_{13}=0$, which means that **B** is also diagonal.

If at most two of the principal values of **T** equal, for example, when $T_1=T_2$, **B** then has the form

$$\mathbf{B}=B_{11}\mathbf{e}_1\otimes\mathbf{e}_1+B_{22}\mathbf{e}_2\otimes\mathbf{e}_2+B_{33}\mathbf{e}_3\otimes\mathbf{e}_3+B_{12}\mathbf{e}_1\otimes\mathbf{e}_2+B_{21}\mathbf{e}_2\otimes\mathbf{e}_1 \tag{10}$$

And since **B** is positive definite,

$$B_1>0, B_3>0, C\equiv B_1B_2-B_{12}{}^2>0 \tag{11}$$

With (8) and (10), (5) gives that:

$$\begin{cases} 0 = \tilde{\psi}_1 - \dfrac{1}{C}\tilde{\psi}_{-1} B_{12} \\ T_{11} = \tilde{\psi}_0 + \tilde{\psi}_1 B_{11} + \dfrac{1}{C}\tilde{\psi}_{-1} B_{22} \\ T_{22} = \tilde{\psi}_0 + \tilde{\psi}_1 B_{22} + \dfrac{1}{C}\tilde{\psi}_{-1} B_{11} \\ T_{33} = \tilde{\psi}_0 + \tilde{\psi}_1 B_{33} + \dfrac{1}{B_{33}}\tilde{\psi}_{-1} \end{cases} \tag{12}$$

Observe that with (7) and (11), the first equation in (12) implies that $B_{12} = 0$. So that when two of the principal values of **T** equal, **B** is still diagonal. And the second and third equations in (12) indicate that $B_{11} = B_{22}$.

Next, we consider the case when all the principal values of **T** equal, i.e. $\mathbf{T} = q\mathbf{I}$, (5) implies that

$$q\mathbf{I} = \tilde{\psi}_0(I_B, II_B, III_B)\mathbf{I} + \tilde{\psi}_1(I_B, II_B, III_B)\mathbf{B} + \tilde{\psi}_{-1}(I_B, II_B, III_B)\mathbf{B}^{-1} \tag{13}$$

Suppose that (13) could be solved for **B**, and we express **B** in its principal coordinate, we have

$$\begin{cases} q = \tilde{\psi}_0 + \tilde{\psi}_1 B_{11} + \dfrac{1}{B_{11}}\tilde{\psi}_{-1} \\ q = \tilde{\psi}_0 + \tilde{\psi}_1 B_{22} + \dfrac{1}{B_{22}}\tilde{\psi}_{-1} \\ q = \tilde{\psi}_0 + \tilde{\psi}_1 B_{33} + \dfrac{1}{B_{33}}\tilde{\psi}_{-1} \end{cases} \tag{14}$$

From (14) we could obtain that $B_{11} = B_{22} = B_{33}$ thus the principal values of **B** are also equal.

If the material is incompressible, we could see that the argument is the same provided that the empirical inequalities hold.

Observe that (*) holds for every compressible or incompressible isotropic elastic material and is the starting point from which at most three component equations could be derived and many universal relations could be obtained.

(*) says that **TB** is symmetric and yields the following three scalar relations for the off diagonal components:

$$\begin{cases} B_{12}(T_{11} - T_{22}) = (B_{11} - B_{22})T_{12} + B_{13}T_{32} - T_{13}B_{32} \\ B_{23}(T_{22} - T_{33}) = (B_{22} - B_{33})T_{23} + B_{21}T_{13} - T_{21}B_{13} \\ B_{31}(T_{33} - T_{11}) = (B_{33} - B_{11})T_{31} + B_{32}T_{21} - T_{32}B_{21} \end{cases} \tag{**}$$

Suppose, if **B** has the following form:

$$\mathbf{B} = B_{11}\mathbf{e}_1 \otimes \mathbf{e}_1 + B_{22}\mathbf{e}_2 \otimes \mathbf{e}_2 + B_{33}\mathbf{e}_3 \otimes \mathbf{e}_3 + B_{12}\mathbf{e}_1 \otimes \mathbf{e}_2 + B_{21}\mathbf{e}_2 \otimes \mathbf{e}_1 \tag{15}$$

T has a corresponding component form.

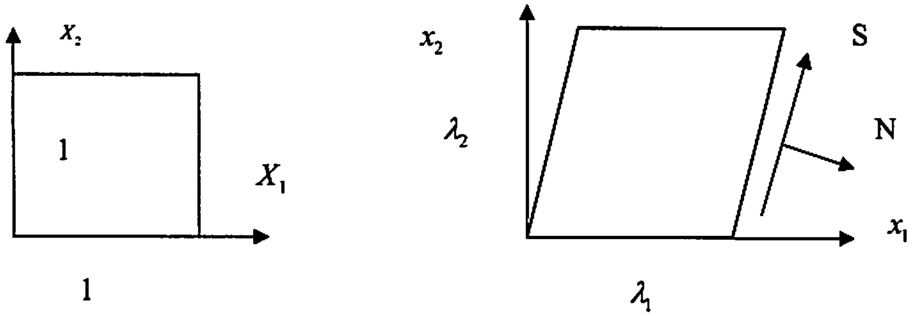
And thus, (**) simplifies to a single universal rule:

$$\frac{T_{11}-T_{22}}{T_{12}}=\frac{B_{11}-B_{22}}{B_{12}} \qquad (***)$$

Beatty [1] has given universal relations for four deformation families. In all the cases, **B** has the form of (15). The cases Beatty has listed are:
Family 1: Bending, stretching and shearing of a rectangular block
Family 2: Straightening, stretching and shearing of a sector of a hollow cylinder
Family 3: Inflation, bending, torsion, extension and shearing of an annular wedge
Family 4: Inflation, bending, extension and azimuthal shearing of an annular wedge

4. **Example** (Beatty[1] and Rajagopal and Wineman [6])

A controlled shear---simple shear superposed on triaxial extension



$$\mathbf{f}(\mathbf{X})=(\lambda_1 X_1+k\lambda_2 X_2)\mathbf{e}_1+\lambda_2 X_2\mathbf{e}_2+\lambda_3 X_3\mathbf{e}_3 \qquad (16)$$

$$[F]=\begin{bmatrix}\lambda_1 & k\lambda_2 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix},\quad [B]=\begin{bmatrix}\lambda_1^2+k^2\lambda_2^2 & k\lambda_2^2 & 0\\ k\lambda_2^2 & \lambda_2^2 & 0\\ 0 & 0 & \lambda_3^2\end{bmatrix} \qquad (17)$$

If the material is incompressible, the following universal relation is possible only when

$$\lambda_1\lambda_2\lambda_3=1 \qquad (18)$$

Now, for every compressible or incompressible, isotropic elastic material, (***) implies

$$T_{11}-T_{22}=\frac{\lambda_1^2+(k^2-1)\lambda_2^2}{k\lambda_2^2}T_{12} \qquad (19)$$

(19) is the universal relation for this deformation.
Notice that when $\lambda_1=\lambda_2=1$, (19) becomes what we have for simple shear:

$$T_{11}-T_{22}=kT_{12} \qquad (20)$$

But $\lambda_3=1$ is not required. So the same universal relation holds for two different deformations of the same compressible material.
The normal and shear tractions on the surface with unit normal $\mathbf{n}=\mathbf{e}_1$ in the reference configuration are:

$$\begin{aligned} N &= \frac{T_{11} + k^2 T_{22} - 2T_{12}k}{1+k^2} \\ S &= \frac{k(T_{11} - T_{22}) + T_{12}(1-k^2)}{1+k^2} \end{aligned} \tag{21}$$

With (19), (20) is written as

$$\begin{aligned} S &= \frac{\lambda_1^2}{\lambda_2^2}\frac{T_{12}}{1+k^2} \\ N &= \frac{AT_{11} + BT_{22}}{1+k^2} \\ \textit{where }\ \mathrm{A} &= \frac{\lambda_1^2 - k^2\lambda_2^2 - \lambda_2^2}{\lambda_1^2 + (k^2-1)\lambda_2^2} \\ \mathrm{B} &= \frac{k^2[\lambda_1^2 + (k^2-1)\lambda_2^2 + 2\lambda_2^2]}{\lambda_1^2 + (k^2-1)\lambda_2^2} \end{aligned} \tag{22}$$

Suppose the normal traction on the surface with unite normal $\mathbf{n} = \mathbf{e}_2$ in the reference configuration vanish, i.e. $T_{22} = 0$

Then,

$$N = \frac{\lambda_1^2 - \lambda_2^2 - k^2\lambda_2^2}{k\lambda_1^2} S \tag{23}$$

Suppose $\mathrm{S} \neq 0$, then N=0 iff

$$\lambda_1^2 = (1+k^2)\lambda_2^2 \tag{24}$$

(24) is another universal relation and it holds for two deformations.

5. **A general scheme for finding linear universal relations** (Bustamante and Ogden [7])

For isotropic materials:

$$\mathbf{T} = \psi_0(I_B, II_B, III_B)\mathbf{I} + \psi_1(I_B, II_B, III_B)\mathbf{B} + \psi_2(I_B, II_B, III_B)\mathbf{B}^2 \tag{25}$$

Write

$$\begin{aligned} &\underset{\sim}{\tau} = (\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6)^T \equiv (T_{11}, T_{22}, T_{33}, T_{32}, T_{31}, T_{21})^T \\ &\mathbf{b}^{(0)} \equiv (1,1,1,0,0,0)^T \\ &\mathbf{b}^{(1)} = (b_1, b_2, b_3, b_4, b_5, b_6)^T \equiv (B_{11}, B_{22}, B_{33}, B_{32}, B_{31}, B_{21})^T \\ &\mathbf{b}^{(1)} = (d_1, d_2, d_3, d_4, d_5, d_6)^T \equiv (d_{11}, d_{22}, d_{33}, d_{32}, d_{31}, d_{21})^T \\ &\textit{where }\ \mathrm{d}_{ij} = (\mathbf{B}^2)_{ij} \end{aligned}$$

Then,

$$\underset{\sim}{\tau} = \psi_\alpha \mathbf{b}^\alpha \qquad \alpha\text{=0,1,2} \tag{26}$$

For linear universal relations, we seek the coefficients a such that

$$\mathbf{a} \cdot \tau = 0 \qquad \Rightarrow \qquad \mathbf{a} \cdot \psi_\alpha \mathbf{b}^\alpha = 0 \tag{27}$$

Note, here **a** and $\underset{\sim}{\tau}$ are vectors in six-dimensional space.

In order that (27) is a universal relation, the coefficients must be independent of ψ_α, α=0,1,2.

Thus,

$$\mathbf{a}\bullet\mathbf{b}^\alpha = 0 \qquad \alpha=0,1,2 \tag{28}$$

(28) could be written as

$$[M]\{a\} = \{0\} \tag{29}$$

Where

$$[M] = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ d_1 & d_2 & d_3 & d_4 & d_5 & d_6 \end{bmatrix} \tag{30}$$

The dimension of the null space of [M] is three. Any solution can be written as linear combinations of vectors in the null space. The following vectors are linearly independent and form a basis for the null space of [M].

$$\begin{cases} \mathbf{a}_1 = \begin{pmatrix} b_6 & -b_6 & 0 & -b_5 & b_4 & b_2 - b_1 \end{pmatrix}^T \\ \mathbf{a}_2 = \begin{pmatrix} b_5 & 0 & -b_5 & -b_6 & b_3 - b_1 & b_4 \end{pmatrix}^T \\ \mathbf{a}_3 = \begin{pmatrix} 0 & b_4 & -b_4 & b_3 - b_2 & -b_6 & b_5 \end{pmatrix}^T \end{cases} \tag{31}$$

Thus,

$$\begin{cases} \tau\bullet\mathbf{a}_1 = B_{12}(T_{11} - T_{22}) - (B_{11} - B_{22})T_{12} - B_{13}T_{32} + T_{13}B_{32} = 0 \\ \tau\bullet\mathbf{a}_2 = B_{31}(T_{33} - T_{11}) - (B_{33} - B_{11})T_{31} - B_{32}T_{21} + T_{32}B_{21} = 0 \\ \tau\bullet\mathbf{a}_3 = B_{23}(T_{22} - T_{33}) - (B_{22} - B_{33})T_{23} - B_{12}T_{13} + T_{12}B_{13} = 0 \end{cases} \tag{32}$$

Thus, any linear universal relation is necessarily a linear combination of the solutions given by (**).

References

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