Asymptotic field at the tip of a crack in a thin sheet of incompressible hyper-elastic material

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1. Motivation

The stress and strain fields at crack tip based on linear elastic theory are well known for various loading conditions. Typically, the stress and strain have a square root singularity at the crack tip ($\sigma \sim K/\sqrt{2\pi r}$), which is in contradiction to the infinitesimal deformation assumption in linear theory. This issue has been resolved by admitting a plastic zone around the crack tip. And with the assumption of small scale yielding, the linear elastic solution is still useful, especially when materials like steel and aluminum are of great interests in fracture mechanics.

However, a physically more reasonable way to investigate the crack tip field is to formulate this problem using large deformation theory. Further more, recently there are more and more interests on soft materials such as polymers and gels. So it is worth studying how large deformation theory changes the crack tip field. Since in large deformation theory there are various constitutive relations and superposition fails, we have to do this study case by case. Here, for simplicity, we consider a crack in a plane stress sheet of *incompressible Neo-Hookean material* under biaxial (symmetric) loading.

2. Nonlinear plane stress equations

2.1 General formulation [1]

Imagine we have a very thin sheet occupying the region $\mathcal{P} = \Omega \times (-h/2, h/2)$, as sketched in Fig.1. The undeformed configuration of the sheet is flat. Here we make three assumptions on the deformation:

a) The deformation is symmetric about the mid-plane $X_3 = 0$, namely that

$$f_3(X_1,X_2,X_3) = -f_3(X_1,X_2,-X_3) \tag{1a}$$

$$f_\alpha(X_1,X_2,X_3) = f_\alpha(X_1,X_2,-X_3)^{*} \tag{1b}$$

b) The up and bottom surfaces are traction free, i.e.,

$$S_{i3} = (X_1,X_2,\pm h/2) = 0, \qquad (X_1,X_2)\in\Omega \tag{2}$$

c) Since the thickness h is much smaller than a characteristic dimension of the mid-plane, we approximatively assume

$$S_{33} = 0, \qquad \text{on } \Omega \tag{3a}$$

$$S_{\alpha\beta,3} = 0 \qquad \text{in } \mathcal{P} \tag{3b}$$

where $,i$ means derivative with respect to X_i.

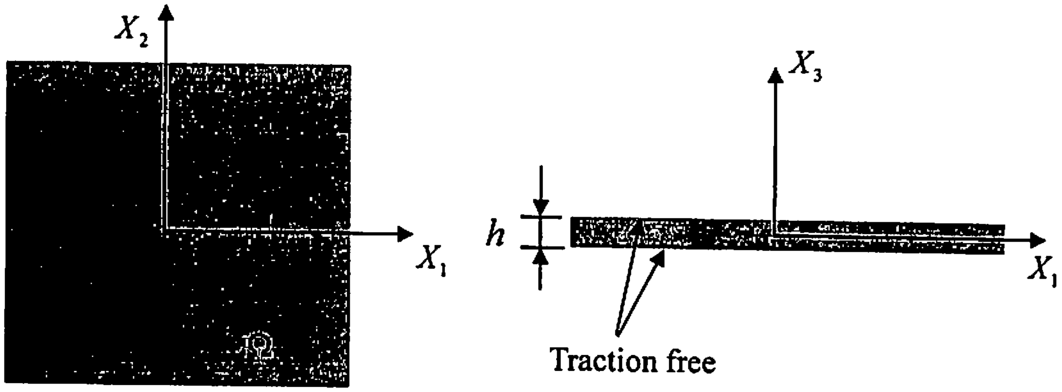

Fig.1 Sketch of the thin sheet in our problem

If g is a function defined in $\mathcal{P}$, then we define the resultant $\tilde{g}$ as

$$\tilde{g} = g(X_1,X_2,X_3 = 0) \tag{4}$$

Therefore, we have

$$\tilde{f}_\alpha(X_1,X_2) = X_\alpha + \tilde{u}_\alpha(X_1,X_2) \ . \tag{5}$$

The equilibrium equation implies

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[*] All the components in this report are with respect to the fixed Cartesian coordinate system $\{\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3\}$. Greek subscripts range over (1,2), while Latin ones can be 1, 2 or 3.

$$\int_{-h/2}^{h/2} (S_{\alpha\beta,\beta} + S_{\alpha 3,3}) dX_3 = 0 \quad . \tag{6}$$

which, because of (3b) and (2), reduces to

$$\tilde{S}_{\alpha\beta,\beta} = 0 \qquad \text{on } \Omega \tag{7}$$

From the assumption (a), it is easy to see that

$$\tilde{f}_{3,\alpha}(X_1, X_2) = 0, \qquad \tilde{f}_{\alpha,3}(X_1, X_2) = 0 . \tag{8}$$

Denote $\lambda(X_1, X_2) = \tilde{f}_{3,3}(X_1, X_2)$, then we have

$$\begin{cases} \tilde{F}_{3\alpha} = \tilde{F}_{\alpha 3} = 0, & \tilde{F}_{33} = \lambda \\ \tilde{B}_{3\alpha} = \tilde{B}_{\alpha 3} = 0, & \tilde{B}_{33} = \lambda^2 \end{cases} , \tag{9}$$

where $\mathbf{B} = \mathbf{F}\mathbf{F}^T$, is the left Cauchy Green strain tensor and λ measures the transverse stretch.

For isotropic homogeneous incompressible hyper-elastic material, the Piola stress is

$$\mathbf{S} = -p\mathbf{F}^{-T} + \phi_0(I_B, II_B)\mathbf{F} + \phi_1(I_B, II_B)\mathbf{B}\mathbf{F} . \tag{10}$$

where $\phi_0 = 2\dfrac{\partial W}{\partial I_B} + 2I_B \dfrac{\partial W}{\partial II_B}$ and $\phi_1 = -2\dfrac{\partial W}{\partial II_B}$.

The Cauchy stress is

$$\mathbf{T} = -p\mathbf{I} + \phi_0(I_B, II_B)\mathbf{B} + \phi_1(I_B, II_B)\mathbf{B}^2 \tag{11}$$

From (9), (10) and (11), it could be shown that

$$\begin{cases} \tilde{S}_{3\alpha} = \tilde{S}_{\alpha 3} = 0, & \tilde{S}_{33} = (\tilde{\phi}_0\lambda^2 + \tilde{\phi}_1\lambda^4 - \tilde{p}) / \lambda \\ \tilde{T}_{3\alpha} = \tilde{T}_{\alpha 3} = 0, & \tilde{T}_{33} = (\tilde{\phi}_0\lambda^2 + \tilde{\phi}_1\lambda^4 - \tilde{p}) \end{cases} \tag{12a}$$

and

$$\tilde{T}_{\alpha\beta} = \tilde{S}_{\alpha\gamma}\tilde{F}_{\beta\gamma} \tag{12b}$$

For example, $S_{31} = -pF_{31}^{-T} + \phi_0 F_{31} + \phi_1 B_{3\alpha} F_{1\alpha} + \phi_1 B_{33} F_{13} = 0$.

By (3a), we have $\tilde{S}_{33} = 0$, which yields

$$\tilde{p} = \tilde{\phi}_0\lambda^2 + \tilde{\phi}_1\lambda^4 \tag{13}$$

If we define

$$J(X_1, X_2) = \det[\tilde{\mathbb{F}}] \tag{14}$$

where $\tilde{\mathbb{F}} = \tilde{F}_{\alpha\beta}\underset{\sim}{e}_\alpha \otimes \underset{\sim}{e}_\beta$. Then by the incompressible condition, we should have

$$\lambda(X_1, X_2) = 1/J(X_1, X_2) \tag{15}$$

From now on, we omit the superscript " ~ " from notations for simplicity. And all the vectors and tensors below are restricted to the mid-plane and thus are two dimensional. Here we summarize what we have done as below.

$$\underset{\sim}{f}(X_1, X_2) = \underset{\sim}{X} + \underset{\sim}{u}(X_1, X_2), \qquad \text{on } \Omega \tag{16}$$

$$\underset{\sim}{\nabla} \bullet \mathbf{S} = \mathbf{0} \qquad \text{on } \Omega \tag{17}$$

$$J = 1/\lambda = \det(\mathbf{F}) \tag{18}$$

$$\mathbf{S} = -p\mathbf{F}^{-T} + \phi_0(I_B, II_B)\mathbf{F} + \phi_1(I_B, II_B)\mathbf{BF} \tag{19a}$$

$$\mathbf{T} = \mathbf{SF}^T \tag{19c}$$

$$p = \phi_0\lambda^2 + \phi_1\lambda^4 \tag{19c}$$

2.2 Our specific problem

Suppose we have an infinitely large sheet of Neo-Hookean material with a crack of length 2a. And we apply biaxial (symmetric) loading at infinity.

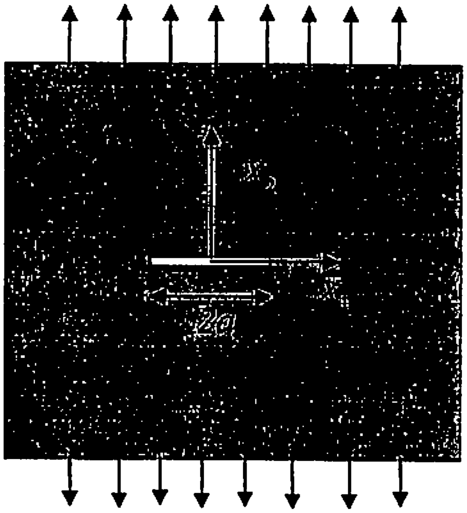

Fig.2 Sketch of the crack in the sheet.

For Neo-Hookean material, the strain energy is

$$W=\frac{\mu}{2}(I_B-3) \tag{20}$$

Therefore, $\phi_0=2\frac{\partial W}{\partial I_B}+2I_B\frac{\partial W}{\partial II_B}=\mu$ and $\phi_1=-2\frac{\partial W}{\partial II_B}=0$. Using (19c), the pressure becomes

$$p=\mu\lambda^2 \tag{21}$$

According to (19a), the 1st Piola stress should be

$$\mathbf{S}=-\mu\lambda^2\mathbf{F}^{-T}+\mu\mathbf{F} \tag{22}$$

The matrix of $\mathbf{F}$ and $\mathbf{F}^{-T}$ with respect to $\{\underset{\sim}{e}_1,\underset{\sim}{e}_2\}$ is

$$[F]=\begin{bmatrix} f_{1,1} & f_{1,2} \\ f_{2,1} & f_{2,2} \end{bmatrix} \tag{23a}$$

$$\left[F^{-T}\right]=\frac{1}{J}\begin{bmatrix} f_{1,1} & -f_{2,1} \\ -f_{1,2} & f_{2,2} \end{bmatrix} \tag{23b}$$

Substitute (22) into the equilibrium equation (17) with (23a,b), we have

$$\nabla^2 f_1=(\lambda^3)_{,1}f_{2,2}-(\lambda^3)_{,2}f_{2,1} \tag{24a}$$

$$\nabla^2 f_2=(\lambda^3)_{,2}f_{1,1}-(\lambda^3)_{,1}f_{1,2} \tag{24b}$$

(24a,b) give a set of nonlinear differential equations for $f_1(X_1,X_2), f_2(X_1,X_2)$. There are two boundary conditions. First, the crack face should be traction free, i.e.,

$$S_{\alpha 2}(|X_1|<a, X_2=0^{\pm})=0 \tag{25}$$

Secondly, the deformation should satisfy the kinematic loading conditions at infinity,

$$\underset{\sim}{f}(\underset{\sim}{X})=\overset{\infty}{\mathbf{F}}\,\underset{\sim}{X}+O(1) \qquad \text{as } X_1^2+X_2^2\to\infty \tag{26}$$

where

$$\left[\overset{\infty}{F}\right]=\begin{bmatrix} \overset{\infty}{\lambda}_1 & 0 \\ 0 & \overset{\infty}{\lambda}_2 \end{bmatrix} \tag{27}$$

(27) implies that we apply a biaxial loading to the sheet at infinity.

3. Asymptotic solution

Theoretically, we can solve (24a,b) subjected to the boundary conditions (25) and (26) and obtain the global stress and strain field for this problem. However, since (24a,b) are nonlinear differential equations, the full field solution is very hard to get. Now, the questions is, can we obtain an asymptotic solution which can sufficiently accurately describe the local stress and strain field near the crack tip?

The answer is yes, at least for this problem. Here we introduce the asymptotic analysis by Geubelle and Knauss [2] in details and then briefly mention the approximate full field solution by Wong and Shield [3].

3.1 Asymptotic analysis [2]

Let's focus on the local region near the right crack tip. Locally, the crack face can be treated as semi-infinite as shown in Fig.3.

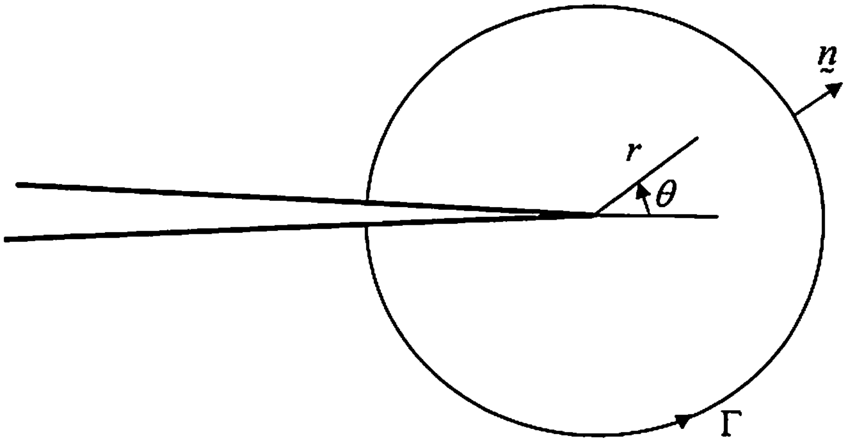

Fig. 3 Geometry of local region near crack tip

We build up a local polar coordinate system (r,θ). Then in this local problem, the traction free boundary condition on crack faces become

$$S_{\alpha 2}(r,\theta=\pm\pi)=0 \tag{28}$$

The far field loading can be expressed as symmetry requirements:

$$f_1(r,\theta)=f_1(r,-\theta) \tag{29a}$$

$$f_2(r,\theta) = -f_2(r,-\theta) \tag{29b}$$

Assume near the crack tip, there exists a separable solution, i.e., we assume the deformation field near the crack tip is

$$\begin{cases} f_1(r,\theta) = r^p v_1(\theta) + o(r^p) \\ f_2(r,\theta) = r^m v_2(\theta) + o(r^m) \end{cases} \quad r \to 0, -\pi \le \theta \le \pi \tag{30}$$

It is required by physical expectations that

$$m, p > 0, \ m \text{ or } p < 1 \tag{31}$$

Physically we expect $p > m$, since the crack opening configuration for $p < m$ (cusp) and $p = m$ (wedge) are all not physically acceptable. Furthermore, let's look at the J-integral around the crack

$$J^* = \int_\Gamma (W n_1 - S_{\alpha\beta} n_\beta f_{\alpha,1}) ds \tag{32}$$

where Γ is any smooth contour around the crack tip and $\underset{\sim}{n}$ is the outward unit normal vector of the contour. Since the J^* is path-independent, we expect that J^* is finite as $r \to 0$. The same argument was used by Hutchison, Rice and Rosengren when they developed the elasto-plastic asymptotic (HRR) field. We then examine the radial dependence of the lowest order terms in (32). According to (20) and (30),

$$W \sim I_B \sim r^{2m-2}, \tag{33}$$

Since $\lambda = 1/J \sim r^{2-m-p}$, by (22) and (23), $S_{\alpha\beta} \sim r^{m-1}$, so

$$S_{\alpha\beta} n_\beta f_{\alpha,1} \sim r^{2m-2} \tag{34}$$

Therefore,

$$(W n_1 - S_{\alpha\beta} n_\beta f_{\alpha,1}) ds \sim r^{2m-2+1} \tag{35}$$

By the argument that J must be finite as $r \to 0$, we conclude that

$$m = 1/2 \tag{36}$$

Now we substitute (30) into the equilibrium equation (24a,b) with $m = 1/2$ and only keep the lowest order terms. With one more assumption that $p < 7/2$, (24a,b) reduce to

$$\nabla^2 f_\alpha = 0 \tag{37}$$

which can also be written as

$$\ddot{v}_1(\theta) + p^2 v_1(\theta) = 0 \tag{38a}$$

$$\ddot{v}_2(\theta) + m^2 v_2(\theta) = 0 \tag{38b}$$

The traction free boundary condition (28) is asymptotically satisfied if

$$\dot{v}_\alpha(\pm\pi) = 0 \tag{39}$$

Derivation of (39) is included in the Appendix.

And the symmetry of the deformation (29a,b) requires

$$\dot{v}_1(0) = 0 \qquad (v_1(\theta) \text{ is even}) \tag{40a}$$

$$v_2(0) = 0 \qquad (v_2(\theta) \text{ is odd}) \tag{40b}$$

The solution of (38a), (38b) subjected to the boundary conditions (39) and (40a,b) is easy to find, that is,

$$v_1(\theta) = C_1 \cos(\theta) \quad \text{with} \quad p = 1 \tag{41a}$$

$$v_2(\theta) = C_2 \sin(\theta/2) \quad \text{with} \quad m = 1/2 \tag{41b}$$

where C_1, C_2 are undetermined constants.

Therefore, the asymptotic deformation field near the crack tip is

$$\begin{cases} f_1 = C_1 r \cos\theta + o(r) \\ f_2 = C_2 r^{1/2} \sin(\theta/2) + o(r^{1/2}) \end{cases} \quad r \to 0, -\pi \le \theta \le \pi \tag{42}$$

The asymptotic 1st Piola stress field is

$$\begin{cases} S_{11} = \mu C_1 \\ S_{12} = 0 \\ S_{21} = -\mu C_2 r^{-1/2} \sin^3(\theta/2) \\ S_{22} = \mu C_2 r^{-1/2} [\sin(\theta/2)\sin\theta + \cos(\theta/2)]/2 \end{cases} \quad r \to 0, -\pi \le \theta \le \pi \tag{43}$$

where C_1, C_2 cannot be determined since we only do a local asymptotic analysis here. To determine C_1, C_2, we should match the asymptotic field to the global solution.

Note: In Geubelle and Knauss's paper, they did the above analysis for a class of incompressible hyper-elastic materials called Generalized Neo-Hookean materials, here only the special case (Neo-Hookean) is introduced.

3.2 Approximate full field solution [3]

Wong and Shield assume that the in-plane deformation is very large across the hole sheet, i.e., $J >> 1$ and $\lambda << 1$ on Ω. A first order approximation is to set $\lambda = 0$ in the equilibrium equation (24a,b) and 1st Piola stress expression (22). We end up with two Laplace equations

$$\nabla^2 f_\alpha = 0 \tag{44}$$

The boundary conditions (25) and (26) become

$$f_{1,2} = 0, f_{2,2} = 0 \qquad \text{at } |X_1| < a, X_2 = 0^\pm \tag{45}$$

$$f_{1,1} = \overset{\infty}{\lambda}_1,\ \ f_{1,2} = 0,\ \ f_{2,1} = 0,\ \ f_{2,2} = \overset{\infty}{\lambda}_2 \quad \text{as } X_1^2 + X_2^2 \to \infty \tag{46}$$

Then solution of (44) subjected to (45) and (46) is found to be

$$f_1 = \overset{\infty}{\lambda}_1 X_1$$

$$f_2 = \frac{\overset{\infty}{\lambda}_2}{\sqrt{2}} sign(X_2) \left\{ a^2 - X_1^2 + X_2^2 + \left[(a^2 - X_1^2 - X_2^2)^2 + 4a^2 X_2^2 \right]^{1/2} \right\}^{1/2} \tag{47}$$

Note that this approximate solution (47) requires very large in-plane deformation J. This solution is more and more accurate as we approach the crack tip, since we expect large deformation near the crack tip. If we extract the asymptotic field from (47), it should agree with that given by (42)

Look at the right crack tip and use the local polar coordinate system (r, θ).

$$X_1 = a + r\cos\theta,\ \ X_2 = r\sin\theta \tag{48}$$

Substitute (48) into (47), we get

$$f_1 = \overset{\infty}{\lambda}_1(a + r\cos\theta)$$
$$f_2 = \frac{\overset{\infty}{\lambda}_2}{\sqrt{2}}\,\mathrm{sign}(\sin\theta)\left\{-2ar\cos\theta - r^2\cos(2\theta) + \left[(2ar\cos\theta + r^2)^2 + 4a^2r^2\sin^2\theta\right]^{1/2}\right\}^{1/2} \tag{49}$$

Neglecting the higher order terms in f_2 as $r \to 0$, (49) becomes

$$\begin{aligned} f_1 &= \overset{\infty}{\lambda}_1\, r\cos\theta + \overset{\infty}{\lambda}_1\, a \\ f_2 &= \overset{\infty}{\lambda}_2\, r^{-1/2}\sin(\theta/2) \end{aligned} \tag{50}$$

which agrees with (42). The constant $\overset{\infty}{\lambda}_1 a$ in f_1 correspond to a translation along $\underset{\sim}{e}_1$, which doesn't affect the stress field.

Appendix: Derivation of (39)

Using (30), we get

$$\begin{cases} f_{1,1} = r^{p-1}v_1(\theta)\cos\theta - r^{p-1}\dot{v}_1(\theta)\sin\theta \\ f_{1,2} = r^{p-1}v_1(\theta)\sin\theta + r^{p-1}\dot{v}_1(\theta)\cos\theta \\ f_{2,1} = r^{m-1}v_2(\theta)\cos\theta - r^{m-1}\dot{v}_2(\theta)\sin\theta \\ f_{1,1} = r^{m-1}v_1(\theta)\sin\theta + r^{m-1}\dot{v}_2(\theta)\cos\theta \end{cases} \tag{A.1}$$

Then

$$J = f_{1,1}f_{2,2} - f_{1,2}f_{2,1} = r^{p+m-2}(v_1\dot{v}_2 - \dot{v}_1 v_2) \tag{A.2}$$

Then $\lambda^2 \sim r^{4-2p-2m}$ is very small as $r \to 0$, so in the expression of 1st Piola stress (22), we assume the first part with $-\mu\lambda^2\mathbf{F}^{-T}$ is much smaller than the second part $\mu\mathbf{F}$, which results

$$S_{12} = \mu r^{p-1}(v_1\sin\theta + \dot{v}_1\cos\theta) \tag{A.3}$$

$$S_{22} = \mu r^{m-1}(v_2\sin\theta + \dot{v}_2\cos\theta) \tag{A.4}$$

The traction boundary conditions $S_{\alpha 2}(r, \theta = \pm\pi) = 0$ yields

$$\dot{v}_\alpha = 0 \tag{A.5}$$

One may find the boundary conditions (A.5) implies that as we approach the traction free boundary, $J \to 0$, $\lambda \to \infty$ and the assumption that the first term with λ^2 in the 1st Piola stress expression (22) can be neglected breaks down. Therefore, one should be aware that (42) is valid in the region near the crack tip but away from the crack face, e.g. region ahead of the crack tip. Wong and Shield noticed this problem and they provide a modified technique to deal with traction free boundary. Please refer to [3] if interested.

Reference

[1] *Knowles, J. K. and Sternberg, E., Journal of Elasticity, 13,257-293, 1983.*

[2] *Geubelle, P. H. and Knauss, W. G, Journal of Elasticity, 35, 61-98,1994.*

[3] *Wong, F. S. and Shield, R. T., ZAMP, 20, 177-199, 1969.*