Axisymmetric Equilibrium Equations for Lipid Membranes

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Contents

1 Abstract

In this article we will use the shell equations we derived in class and specialize it to axisymmetric equations for a shell with unit director [Ant]. In the process we will be introducing new strain variables. The next step is to

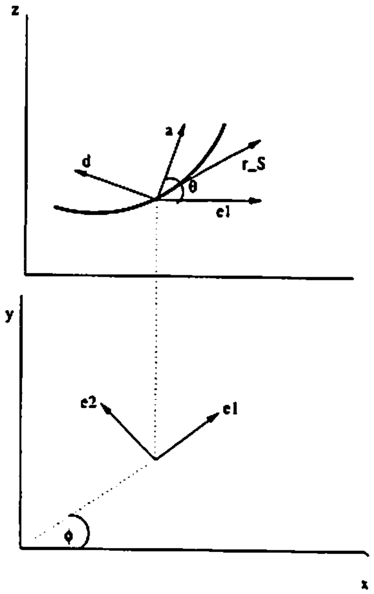

Figure 1:

derive equations of equilibrium for a spherical vesicle. For this, we will cast the Hilfrisch's free energy [Hil] (a special case of the free energy that Philip [Phil] derived in class) in terms of the new strain variables. We plug in this free energy into the axisymmetric shell equations and get an axisymmetric version of the equilibrium equations derived by Jenkins[1] [Jen]

2 Axisymmetric Equations for a Shell

Let us derive the axisymmetric equations for a shell assuming a unit director. We have seen in class that under this assumption and with material objectivity we only need the linear and angular momentum balance as our governing equations. Before starting to derive the equilibrium equations, let us set up a convenient coordinate system and establish the following definitions:

$\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ Denote a fixed orthonormal basis for $\mathbb{E}^3$.

$\mathbf{S} := (S, \Phi)$ Where S is arclength and Φ is the azimuthal angle in the reference configuration.

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[1] Jenkins' version is formulated in the current configurations using variational principles

$\mathbf{e}_1(\Phi) := \cos\Phi\mathbf{i} + \sin\Phi\mathbf{j}$

$\mathbf{e}_2(\Phi) := -\sin\Phi\mathbf{i} + \cos\Phi\mathbf{j}$

$\mathbf{e}_3 := \mathbf{k}$

$\mathbf{r}^o(S,\Phi) := r^o(S)\mathbf{e}_1(\Phi) + z^o\mathbf{k}$ is the position vector in the reference configuration.

$\mathbf{a}^o := \cos\theta^o(S)\mathbf{e}_1(\Phi) + \sin\theta^o(S)\mathbf{k}$ where θ^o is as indicated in Fig. 2

$\mathbf{d}^o := \mathbf{a}_o \times \mathbf{e}_2$ is the director of the reference configuration.

$\mathbf{r}(S,\Phi) := r(S)\mathbf{e}_1(\Phi) + z\mathbf{k}$ is the position vector in the current configuration.

$\mathbf{a} := \cos\theta(S)\mathbf{e}_1(\Phi) + \sin\theta(S)\mathbf{k}$

$\mathbf{d} := \mathbf{a} \times \mathbf{e}_2$ is the director of the current configuration.

$\nu := r'(S)\cos\theta + z'(S)\sin\theta$ which is a measure of stretch along the arc lenth.

$\eta := z'(S)\cos\theta - r'(S)\sin\theta$ which is a measure of shear.

Subscripts in what follows denotes partial derivatives with respect to appropriate variables

With the above defintions we have

$\mathbf{r}_S = \nu\mathbf{a} + \eta\mathbf{d}$

$\mathbf{r}_\Phi = r(S)\mathbf{e}_2(\Phi)$

$\mathbf{d}_S = -\theta'(S)\mathbf{a}(S,\Phi)$

$\mathbf{d}_\Phi = -\sin\theta(S)\mathbf{e}_2(\Phi)$

With this let is introduce the following strain variables

$$\tau := \frac{r(S)}{r^o(S)}, \sigma := \frac{\sin\theta(S)}{r^o(S)}, \mu = \theta'(S) \tag{1}$$

here τ is the stretch along the azimuthal direction and μ is a measure of bending. To get a feel for σ, imagine deforming a flat circular plate into a cone. Let us further assume unshearablity along the thickness (that is, the director is the normal). Then, althought $\mu = 0$, σ is not equal to zero. Thus, bending out of plane is captured by σ.

2.1 Linear Momentum Balance

Recall that our equation for linear momentum balance for a shell with body force b per unit area of a *flat* reference configuration derived in class was

$$(\mathbf{N}_\alpha)_\alpha + \mathbf{b} = 0 \tag{2}$$

But if the reference configuration is curved, then the above equation is to modified as follows:

$$(\sqrt{A})^{-1}(\sqrt{A}\mathbf{N}_\alpha)_\alpha + \mathbf{b} = 0 \tag{3}$$

where $\alpha \in \{S, \Phi\}$ and $\sqrt{A} = r_0(S)$. Let me sketch the proof for (3). Recall from that (2) comes by integration of parts of the following equation

$$\int_\Omega \mathbf{N}_\alpha \xi_\alpha + \mathbf{b}\xi \ \ dA = 0 \tag{4}$$

where ξ is the variation of the position and Ω is the surface in the reference configuration. On a surface parametrized by generalized coordinates $\{S, \Phi\}$, area element, $dA = \sqrt{A} dS d\Phi$[2]. Where $\sqrt{A}$ is defined in Section (4.1) and in general depends on $\{S, \Phi\}$. So, the above equation becomes,

$$\int_\Omega (\mathbf{N}_\alpha \xi_\alpha + \mathbf{b}\xi) \ \ \sqrt{A} dS d\Phi = 0 \tag{5}$$

And now integrating by parts,

$$\int_\Omega (\sqrt{A}\mathbf{N}_\alpha)_\alpha \xi + \sqrt{A}\mathbf{b}\xi \ \ dS d\Phi = 0 \tag{6}$$

Which then by localization argument gives us (3).

If we, furthermore, assume hyperelasticity and postulate a free energy $\tilde{\mathcal{E}}(\mathbf{r}_S, \mathbf{r}_\Phi, \mathbf{d}, \mathbf{d}_S, \mathbf{d}_\Phi)$ then we saw in class that

$$\mathbf{N}_S = \frac{\partial \tilde{\mathcal{E}}}{\partial \mathbf{r}_S} \tag{7}$$

$$\mathbf{N}_\Phi = \frac{\partial \tilde{\mathcal{E}}}{\partial \mathbf{r}_\Phi} \tag{8}$$

$$\tag{9}$$

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[2] For example, in polar coordinates parameterized by $\{r, \theta\}$, $dA = r dr d\theta$

We shall now express these equations in terms of the strain variables $(\nu, \eta, \tau, \sigma, \mu)$ we have defined above. Let's assume $\mathcal{E}(\nu, \eta, \tau, \sigma, \mu) := \tilde{\mathcal{E}}(\mathbf{r}_S, \mathbf{r}_\Phi, \mathbf{d}, \mathbf{d}_S, \mathbf{d}_\Phi)$. We then have,

$$\mathbf{N}_S = \frac{\partial \mathcal{E}}{\partial \nu}\mathbf{a} + \frac{\partial \mathcal{E}}{\eta}\mathbf{d} =: N(S)\mathbf{a} + H(S)\mathbf{d} \tag{10}$$

$$\mathbf{N}_\Phi = \frac{\partial \mathcal{E}}{\partial r(S)}\mathbf{e}_2 = \frac{1}{r^o(S)}\frac{\partial \mathcal{E}}{\partial \tau}\mathbf{e}_2 =: \frac{1}{r^o}T(S)\mathbf{e}_2 \tag{11}$$

So, $N = \frac{\partial \mathcal{E}}{\partial \nu}$, $H = \frac{\partial \mathcal{E}}{\partial \eta}$ and $T = \frac{\partial \mathcal{E}}{\partial \tau}$. Let us now write the linear momentum balance (3) using the above defintions as

$$(r^o(N\mathbf{a} + H\mathbf{d}))_S + (T(S)\mathbf{e}_2)_\Phi + r^o\mathbf{b} = 0 \tag{12}$$

That is

$$(r^o(N\mathbf{a} + H\mathbf{d}))_S - (T(S)\mathbf{e}_1 + r^o\mathbf{b} = 0 \tag{13}$$

2.2 Angular Momentum Balance

The angular momentum balance for a shell with a body couple per unit area $\mathbf{l}$ derived in class with the argument given in Sec. (2.1) becomes

$$(\sqrt{A})^{-1}(\sqrt{A}\mathbf{M}_\alpha)_\alpha + \mathbf{r}_{\alpha\cdot} \times \mathbf{N}_\alpha + \mathbf{l} = 0 \tag{14}$$

Just as in linear momentum balance, we shall put the above equation in terms of the strain variables defined in (1). The following equations follow from the free energy $\mathcal{E}$

$$\mathbf{M}_S = \mathbf{d} \times \frac{\partial \tilde{\mathcal{E}}}{\partial \mathbf{d}_S} \tag{15}$$

$$\mathbf{M}_\Phi = \mathbf{d} \times \frac{\partial \tilde{\mathcal{E}}}{\partial \mathbf{d}_\Phi} \tag{16}$$

Using the identities $\mathbf{d} \times \mathbf{a} = \mathbf{e}_2$, $\mathbf{d} \times \mathbf{e}_2 = -\mathbf{a}$ the above equations become

$$\mathbf{M}_S = \mathbf{d} \times (-\frac{\partial \mathcal{E}}{\partial \mu}\mathbf{a}) = -\frac{\partial \mathcal{E}}{\partial \mu}\mathbf{e}_2 =: -M(S)\mathbf{e}_2 \tag{17}$$

$$\mathbf{M}_\Phi = \mathbf{d} \times (-\frac{1}{r^o}\frac{\partial \mathcal{E}}{\partial \sigma}\mathbf{e}_2) = \frac{1}{r^o}\frac{\partial \mathcal{E}}{\partial \sigma}\mathbf{a} =: \frac{1}{r^o}\Sigma(S)\mathbf{a} \tag{18}$$

That is, we have defined $\frac{\partial \mathcal{E}}{\partial \mu} = M$ and $\frac{\partial \mathcal{E}}{\partial \sigma} = \Sigma$. With these, (14) becomes

$$(-r^o M(S)\mathbf{e}_2(\Phi))_S + (\Sigma(S)\mathbf{a}(S,\Phi))_\Phi + r^o[(\nu \mathbf{a} + \eta \mathbf{d}) \times (\mathbf{N}\mathbf{a} + H\mathbf{d}) \tag{19}$$

$$+\frac{r}{r^o} T\mathbf{e}_2 \times \mathbf{e}_2] + r^o \mathbf{l} = \mathbf{0} \tag{20}$$

Noting that $\frac{\partial \mathbf{a}}{\partial \Phi} = \cos\theta(S)\mathbf{e}_2$ and $\mathbf{d} \times \mathbf{a} = \mathbf{e}_2$ and dotting the resulting equation with $\mathbf{e}_2$ we get

$$(r^o M)_S - \Sigma \cos\theta(S) + r^o(\nu H - \eta N) - r^o \mathbf{l} \cdot \mathbf{e}_2 = 0 \tag{21}$$

3 Equilibrium Equations for an Spherical Axisymmetric Shell

Let us consider a shell whose reference configuration is spherical with a unit radius pressurized from the inside with pressure p. We will also assume zero body couple ($\mathbf{l} = =0$). For such a shell $r^o(S) = \sin S$, $z^o(S) = \cos S$, $\theta^o(S) = S$ and $\mathbf{b} = p(-\eta \mathbf{a} + \nu \mathbf{d}) r(S)/r^o(S)$. The linear momentum balance (13) then becomes

$$(\sin S(N\mathbf{a} + H\mathbf{d}))_S - T\mathbf{e}_1 + r(S)p(-\eta \mathbf{a} + \nu \mathbf{d}) = 0 \tag{22}$$

Using $\mathbf{a}_S = \theta'(S)\mathbf{d}$, $\mathbf{d}_S = -\theta'(S)\mathbf{a}$ and $\mathbf{e}_1 = \cos\theta \mathbf{a} - \sin\theta \mathbf{d}$

$$((\sin SN)_S - \sin SH\theta'(S) - T\cos\theta - r(S)p\eta)\mathbf{a} + \tag{23}$$

$$((\sin SH)_S + \sin SN\theta'(S) + \sin\theta T + r(S)p\nu)\mathbf{d} = 0 \tag{24}$$

The angular momentum balance (21) becomes

$$(\sin SM)_S - \Sigma\cos\theta + \sin S[\nu H - \eta N] = 0 \tag{25}$$

Summarizing equations (23) and (25)

$$((\sin SN)_S - \sin SH\theta'(S) - T\cos\theta - r(S)p\eta) = 0 \tag{26}$$

$$((\sin SH)_S + \sin SN\theta'(S) + \sin\theta T + r(S)p\nu) = 0 \tag{27}$$

$$(\sin SM)_S - \Sigma\cos\theta + \sin S[\nu H - \eta N] = 0 \tag{28}$$

4 Lipid Membranes

We have seen in Philip's talk that the lipid membranes have free energy which depends on h, the mean curvature and k, the gaussian curvature. We will assume that the lipid bilayer is incompressible (that is locally area preserving) and specialize this constitutive law to our axisymmetric problem. The assumptions involved are:

1. Lipid is unshearable along the thickness. That is the director is normal to the surface, $\mathbf{d} = \mathbf{n}$ which implies that $\eta = 0$.
2. Locally area preserving. That is $\nu\tau = 1$.
3. Lipid has an energy function $\tilde{\Upsilon}(h,k) = D_1 h^2 + D_2 k$

To do this we need to express h and k in terms of our strain variables $(\nu, \eta, \tau, \sigma, \mu)$. So, let us express the first and second fundamental forms $\mathbf{C}$ and κ in terms of the strain variables. Recall that the first fundamental form is defined as $\mathbf{C} = [C_{\alpha,\beta}] = [\mathbf{r}_\alpha \cdot \mathbf{r}_\beta]$ and the second fundamental form is defined as $\kappa = [\kappa_{\alpha\beta}] = [\mathbf{r}_{\alpha\beta} \cdot \mathbf{d}]$ where $\alpha, \beta \in \{S, \Phi\}$

4.1 The First Fundamental Form

Since $\mathbf{r}_S = \nu\mathbf{a} + \eta\mathbf{d}$ and $\mathbf{r}_\Phi = r\mathbf{e}_2$.

$$[C] = \begin{pmatrix} \mathbf{r}_S \cdot \mathbf{r}_S & \mathbf{r}_S \cdot \mathbf{r}_\Phi \\ \mathbf{r}_\Phi \cdot \mathbf{r}_S & \mathbf{r}_\Phi \cdot \mathbf{r}_\Phi \end{pmatrix} \tag{29}$$

$$[C] = \begin{pmatrix} \nu^2 + \eta^2 & 0 \\ 0 & r^2 \end{pmatrix} \tag{30}$$

Since $\mathbf{a}\cdot\mathbf{a} = \mathbf{d}\cdot\mathbf{d} = \mathbf{e}_2\cdot\mathbf{e}_2 = 1$ and $\mathbf{a}\cdot\mathbf{d} = \mathbf{d}\cdot\mathbf{e}_2 = \mathbf{e}_2\cdot\mathbf{a} = 0$. Let $[C]^o$ represent the components of the first fundamental form of the reference configuration, then $\nu^o = 1, \eta^o = 0$ and $[C]^o = \begin{pmatrix} 1 & 0 \\ 0 & r^{o2} \end{pmatrix}$ and so $det([C]^o) = r^{o2}$. Thus,

$$J := \sqrt{det([C])/det([C]^o)} = \sqrt{\nu^2 + \eta^2}\, r/r^o = \sqrt{\nu^2 + \eta^2}\,\tau \tag{31}$$

In, particular, with zero shear, i.e, $\eta = 0$, $J = \nu\tau$. This equation will be used later.

4.2 The Second Fundamental Form

$$\mathbf{r}_{SS} = \nu'(S)\mathbf{a}(S,\Phi) + \nu(S)\mathbf{a}_S(S,\Phi) = \nu'\mathbf{a} + \nu\theta'\mathbf{d} \tag{32}$$

$$\mathbf{r}_{\Phi\Phi} = r(S)\mathbf{e}_2(\Phi)_\Phi = -r(S)\mathbf{e}_1(\Phi) \tag{33}$$

$$\mathbf{r}_{\Phi S} = r'(S)\mathbf{e}_2(\Phi) \tag{34}$$

Thus,

$$[\kappa] = -\begin{pmatrix} \mathbf{r}_{SS}\cdot\mathbf{d} & \mathbf{r}_{S\Phi}\cdot\mathbf{d} \\ \mathbf{r}_{S\Phi}\cdot\mathbf{d} & \mathbf{r}_{\Phi\Phi}\cdot\mathbf{d} \end{pmatrix} \tag{35}$$

$$= -\begin{pmatrix} \nu\theta' & 0 \\ 0 & r(S)\sin\theta \end{pmatrix} \tag{36}$$

where we have used $\mathbf{d}\cdot\mathbf{e}_2 = (\mathbf{a}\times\mathbf{e}_2)\cdot\mathbf{e}_2 = 0$ and $\mathbf{d}\cdot\mathbf{e}_1 = \mathbf{a}\cdot\mathbf{e}_2\times\mathbf{e}_1 = -\mathbf{a}\cdot\mathbf{k} = -\sin\theta$.

4.3 Curvature Tensor

The components of the curvature tensor is defined as $[L] = [C]^{-1}[\kappa]$.

$$[L] = -\begin{pmatrix} \frac{1}{\nu^2} & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}\begin{pmatrix} \nu\theta' & 0 \\ 0 & r\sin\theta \end{pmatrix} \tag{37}$$

$$= -\begin{pmatrix} \frac{\theta'}{\nu} & 0 \\ 0 & \frac{\sin\theta}{r} \end{pmatrix} \tag{38}$$

The compents of the curvature tensor in terms of our strain variables are

$$[L] = -\begin{pmatrix} \frac{\mu}{\nu} & 0 \\ 0 & \frac{\sigma}{\tau} \end{pmatrix} \tag{39}$$

The mean curvature (h) and the gaussian curvature (k) are given by

$$h = -\frac{1}{2}\left(\frac{\mu}{\nu} + \frac{\sigma}{\tau}\right) \tag{40}$$

$$k = \frac{\mu\sigma}{\nu\tau} \tag{41}$$

4.4 The Constitutive Law

The free energy for the lipid membranes stated in the beginning of the section include the constraint of local area preservation, *i.e.* $J = \nu\tau = 1$, our free energy with the constraint added becomes

$$\Upsilon(\nu, \eta, \tau, \sigma, \mu) = \frac{1}{4}D_1\left(\frac{\mu}{\nu} + \frac{\sigma}{\tau}\right)^2 + D_2\frac{\mu\sigma}{\nu\tau} + \gamma(\nu\tau - 1) \tag{42}$$

Which may be simplified to

$$\Upsilon(\nu, \eta, \tau, \sigma, \mu) = \frac{1}{4}D_1\left(\frac{\mu}{\nu} + \frac{\sigma}{\tau}\right)^2 + D_2\mu\sigma + \gamma(\nu\tau - 1) \tag{43}$$

because $\nu\tau = 1$. Here, γ is a lagrange multiplier which in general could depend on S. The above energy leads us to the following set of constitutive equations

$$N = \frac{\partial\Upsilon}{\partial\nu} = -\frac{1}{2}D_1\left(\frac{\mu}{\nu} + \sigma\nu\right)\frac{\mu}{\nu^2} + \gamma\tau - D_2\frac{\mu\sigma}{\nu^2\tau} \tag{44}$$

$$T = \frac{\partial\Upsilon}{\partial\tau} = -\frac{1}{2}D_1\left(\frac{\mu}{\nu} + \sigma\nu\right)\sigma\nu^2 + \gamma\nu - D_2\frac{\mu\sigma}{\nu\tau^2} \tag{45}$$

$$\Sigma = \frac{\partial\Upsilon}{\partial\sigma} = \frac{1}{2}D_1\left(\frac{\mu}{\nu} + \sigma\nu\right)\nu + D_2\mu \tag{46}$$

$$M = \frac{\partial\Upsilon}{\partial\mu} = \frac{1}{2}D_1\left(\frac{\mu}{\nu} + \sigma\nu\right)\tau + D_2\sigma \tag{47}$$

We shall express our equilibrium equations in terms of the mean curvature and Gaussian curvature, and so let us write the above equtions as

$$N = D_1 h\frac{\mu}{\nu^2} + \gamma\tau - D_2\frac{k}{\nu} \tag{48}$$

$$T = D_1 h\sigma\nu^2 + \gamma\nu - D_2\frac{k}{\tau} \tag{49}$$

$$\Sigma = -D_1 h\nu + D_2\mu \tag{50}$$

$$M = -D_1 h\tau + D_2\sigma \tag{51}$$

Where we have used $\nu\tau = 1$ in the last terms of the first two equations. Our assumption that the membrane is unshearable along it's thickness means that **d** is the normal to the surface and $\eta = 0$. This makes H another lagrange multiplier.

5 Equilibrium Equations for an Axisymmetric Vesicle

Plug in expressions for M and Σ, (48) into (28)

$$-\sin S\nu H = \sin S(M(S))_S + \cos SM - \Sigma\cos\theta \tag{52}$$

$$= [\sin S\,(-D_1 h\tau + D_2\sigma)_S + \cos S\,(-D_1 h\tau + D_2\sigma) - (-D_1 h\nu + D_2\mu)\cos\theta]$$

$$= -D_1 \sin S\tau h_S \tag{53}$$

Where we have used results (65) and (66) from the Appendix to get to the last equality. Thus,

$$H = D_1\tau^2 h_S \tag{54}$$

Plug in expressions for T and N (48), and for H (54) into (28),

$$\left[\sin SD_1\tau^2 h_S\right]_S + \sin S\theta'\left(D_1 h\mu\tau^2 + \gamma\tau - D_2\frac{k}{\nu}\right) + \sin\theta\left(D_1 h\sigma\nu^2 + \gamma\nu - D_2\frac{k}{\tau}\right) + pr\nu = 0$$

Define $Q := D_1\tau^2 h_S = \frac{D_1}{\nu^2}h_S$ and use $\theta' = \mu$, $\sin\theta = \sigma\sin S$, $r = \tau\sin S$ to get

$$(\sin SQ_S + Q\cos S) + \sin S\left[D_1 h(\mu^2\tau^2 + \sigma^2\nu^2) + \gamma(\mu\tau + \sigma\nu) - D_2 k\left(\frac{\mu}{\nu} + \frac{\sigma}{\tau}\right)\right] + \sin S\tau\nu p = 0 \tag{55}$$

Let us now use $\tau\nu = 1$,

$$(Q_S + Q\cot S) + \left[D_1 h\left(\frac{\mu^2}{\nu^2} + \frac{\sigma^2}{\tau^2}\right) + \gamma\left(\frac{\mu}{\nu} + \frac{\sigma}{\tau}\right)\right] + p = 0 \tag{56}$$

But $\left(\frac{\mu}{\nu} + \frac{\sigma}{\tau}\right) = -2h$ and $\left(\frac{\mu^2}{\nu^2} + \frac{\sigma^2}{\tau^2}\right) = \left(\frac{\mu}{\nu} + \frac{\sigma}{\tau}\right)^2 - 2\frac{\mu\sigma}{\nu\tau} = 4h^2 - 2k$. So,

$$(Q_S + Q\cot S) + 2D_1 h(2h^2 - k) + 2D_2 kh - 2\gamma h + p = 0 \tag{57}$$

The last of the equilibrium equations (26),

$$(\sin S(D_1 h\tau^2\mu + \gamma\tau - D_2 k\tau))_S - \sin S\mu D_1\tau^2 h_S - (D_1 h\sigma\nu^2 + \gamma\nu - D_2 k\nu)\cos\theta = 0 \tag{58}$$

With $r' = (\tau\sin S)_S = \nu\cos\theta$, one of the term involving γ cancels. Expanding $(\sin SD_1 h\tau^2\mu)_S$ and using (74), we get

$$D_1(\sin S\tau^2\mu)_S h - D_1 h\sigma\nu^2\cos\theta + \sin S\gamma_S\tau - D_2\sin S\tau k_S = 0 \tag{59}$$

$$D_1 h\left((\sin S\tau^2\mu)_S - \sigma\nu^2\cos\theta\right) + \sin S\gamma_S\tau - D_2\sin S\tau k_S = 0 \tag{60}$$

Now, using (71),

$$-2D_1\tau\sin Shh_S + \tau\sin S\gamma_S - D_2\sin S\tau k_S = 0 \tag{61}$$

which is can be written as

$$\tau\sin S(D_1h^2 + D_2k - \gamma)_S = 0 \tag{62}$$

That is,

$$D_1h^2 + D_2k - \gamma = d \tag{63}$$

where d is a constant. Putting (57) and (63) together we have,

$$(Q_S + Q\cot S) + 2D_1h((h^2 - k) + d) + p = 0 \tag{64}$$

Notice that D_2, the gaussian curvature does not enter the equations. Which in [Jen] falls out by the Gauss-Bonnet theorem. The above equation is obtained in [Sov] by a different method.

6 Appendix

1.
$$\sin S\tau_S + \cos S\tau = \nu\cos\theta \tag{65}$$

Pf: Since $\sin S\tau = r, \sin S\tau_S + \cos S\tau = r_S = \nu\cos\theta$

2.
$$\sin S\sigma_S + \cos S\sigma = \mu\cos\theta \tag{66}$$

Pf: Since $\sin S\sigma = \sin\theta$, $\sin S\sigma_S + \cos S\sigma = \cos\theta\theta' = \mu\cos\theta$

3.
$$2h_S = -\frac{(\sin S\mu\tau^2)_S - \sigma\nu^2\cos\theta}{\tau\sin S} \tag{67}$$

Pf:

$$-2h_S = (\mu/\nu + \sigma/\tau)_S = \left(\frac{(\mu\tau^2 + \sigma)}{\tau}\right)_S \tag{68}$$

$$= \left(\frac{\sin S(\mu\tau^2+\sigma)}{r}\right)_S \quad \textit{Since} \quad \tau = r/\sin S \tag{69}$$

$$= \frac{r[(\sin S\mu\tau^2)_S + (\sin S\sigma)_S] - \sin S(\mu\tau^2+\sigma)\nu\cos\theta}{r^2} \tag{70}$$

where we have substituted $r' = \nu\cos\theta$. If we now use $\sin S = r/\tau = r\nu$ in the last term,

$$-2h_S = \frac{(\sin S\mu\tau^2)_S + (\sin S\sigma)_S - (\mu\tau^2+\sigma)\nu^2\cos\theta}{r} \tag{71}$$

Finally $(\sigma\sin S) = \sin\theta$ implies, $(\sigma\sin S)_S = \theta'\cos\theta = \mu\cos\theta)$. This along with $\nu\tau = 1$ cancels $(\sigma\sin S)_S$ with the last term in the previous expression and we get

$$-2h_S = \frac{(\sin S\mu\tau^2)_S - \sigma\nu^2\cos\theta}{r} \tag{72}$$

which may be simplied, using $r = \sin S\tau$ to

$$2h_S = -\frac{(\sin S\mu\tau^2)_S - \sigma\nu^2\cos\theta}{\tau\sin S} \tag{73}$$

4.

$$(\sin Sk\tau)_S = k\nu\cos\theta + \sin S\tau k_S \tag{74}$$

Pf:

$$(\sin Sk\tau)_S = k(\cos S\tau + \sin S\tau_S) + \sin S\tau k_S \tag{75}$$

$$= k(\sin\tau)_S + \sin S\tau k_S \tag{76}$$

Note that $\tau r^o = \tau\sin S = r$ and since $\eta = z'\cos\theta - r'\sin\theta = 0$ and $\nu = r'\cos\theta + z'\sin\theta$ we have $\nu\cos\theta = r'$. So, substituting $r' = \nu\cos\theta$,

$$(\sin Sk\tau)_S = k\nu\cos\theta + \sin S\tau k_S \tag{77}$$

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