Lecture 15

where $\underset{\sim}{H}_o(\mathcal{P}) = \int_{\mathcal{P}} \underset{\sim}{f} \times \frac{\partial \underset{\sim}{f}}{\partial t} \rho_o \, dV$

$\int_{\partial \mathcal{P}} \underset{\sim}{f} \times \underset{\sim}{S} \underset{\sim}{m} \, ds =$ (3 parts)

$$\int_{\Omega} (\underset{\sim}{r} + \Xi_\alpha \underset{\sim}{d}_\alpha) \times \underset{\approx}{S} \underset{\sim}{e}_3 \, dA \Big|_{s_1}^{s_2}$$

$$+ \int_{s_1}^{s_2} \oint_{\partial\Omega} (\underset{\sim}{r} + \Xi_\alpha \underset{\sim}{d}_\alpha) \times \underset{\sim}{\hat{s}} \, dl \, ds$$

$$= \overset{(p.124)}{(\underset{\sim}{r} \times \underset{\sim}{n})} \Big|_{s_1}^{s_2} + \left(\int_{\Omega} \Xi_\alpha \underset{\sim}{d}_\alpha \times \underset{\approx}{S} \underset{\sim}{e}_3 \, dA \right) \Big|_{s_1}^{s_2}$$

$$+ \int_{s_1}^{s_2} \underset{\sim}{r} \times \oint_{\partial\Omega} \underset{\sim}{\hat{s}} \, dl \, ds + \int_{s_1}^{s_2} \oint_{\partial\Omega} \Xi_\alpha \underset{\sim}{d}_\alpha \times \underset{\sim}{\hat{s}} \, dl \, ds$$

$$\int_{\mathcal{P}} \underset{\sim}{f} \times \underset{\sim}{\hat{b}} \, dV = \int_{s_1}^{s_2} \int_{\Omega} (\underset{\sim}{r} + \Xi_\alpha \underset{\sim}{d}_\alpha) \times \underset{\sim}{\hat{b}} \, dA \, ds$$

$$= \int_{s_1}^{s_2} \underset{\sim}{r} \times \int_{\Omega} \underset{\sim}{\hat{b}} \, dA \, ds + \int_{s_1}^{s_2} \int_{\Omega} \Xi_\alpha \underset{\sim}{d}_\alpha \times \underset{\sim}{\hat{b}} \, dA \, ds.$$

$$\therefore \underset{\sim}{M}_o(\mathcal{P}) = (\underset{\sim}{r} \times \underset{\sim}{n}) \Big|_{s_1}^{s_2} + \left(\int_{\Omega} \Xi_\alpha \underset{\sim}{d}_\alpha \times \underset{\approx}{S} \underset{\sim}{e}_3 \, dA \right) \Big|_{s_1}^{s_2}$$

$$+ \int_{s_1}^{s_2} \underset{\sim}{r} \times \overset{(p.125)}{\underset{\sim}{\hat{b}}(s,t)} \, ds +$$ (over)

$$\int_{s_1}^{s_2}\left[\oint_{\partial\Omega} X_\alpha \mathbf{d}_\alpha \times \hat{\mathbf{s}}\, dl + \int_\Omega X_\alpha \mathbf{d}_\alpha \times \hat{\mathbf{b}}\, dA\right] ds$$

Define

$$\mathbf{m}(s,t) \equiv \int_\Omega X_\alpha \mathbf{d}_\alpha \times \mathbf{S}(X_1, X_2, s, t)\mathbf{e}_3\, dA$$

contact couple

$$\hat{\mathbf{g}}(s,t) \equiv [\quad]$$ body couple (from external fields)

Then
$$\mathbf{M}_0(s_1, s_2) = \left(\mathbf{r} \times \mathbf{n} + \mathbf{m}\right)\Big|_{s_1}^{s_2} + \int_{s_1}^{s_2}\left(\mathbf{r} \times \hat{\mathbf{b}} + \hat{\mathbf{g}}\right) ds$$

$$\mathbf{H}_0(\mathcal{B}) = \int_{s_1}^{s_2}\int_\Omega (\mathbf{r} + X_\alpha \mathbf{d}_\alpha) \times (\mathbf{r}_t + X_\beta \mathbf{d}_{\beta,t})\rho_0\, dA\, ds$$

centroidal

$$= \int_{s_1}^{s_2} (\mathbf{r} \times \mathbf{r}_t)\overbrace{\left(\int_\Omega \rho_0\, dA\right)}^{\rho_0(s)}$$

$$+ \int_{s_1}^{s_2} \sum_{\alpha=1}^{2} \mathbf{d}_\alpha \times \mathbf{d}_{\alpha,t}\left(\int_\Omega X_\alpha^2 \rho_0\, dA\right) ds$$

Define
$$J_\alpha(s) \equiv \int_\Omega X_\alpha^2 \rho_0\, dA$$

mass moment of inertia of cross section

$$\therefore \quad \mathbf{H}_0(s_1, s_2) = \int_{s_1}^{s_2} \Big[\mathbf{r} \times \rho_0 \mathbf{r}_t + \sum_{\alpha=1}^{2} J_\alpha \mathbf{d}_\alpha \times \mathbf{d}_{\alpha,t} \Big] ds$$

AMB $\quad \mathbf{M}_0(s_1, s_2) = \dfrac{d}{dt} \mathbf{H}_0(s_1, s_2)$

LMB

$$\Rightarrow \quad (\mathbf{r} \times \mathbf{n} + \mathbf{m}) \Big|_{s_1}^{s_2} + \int_{s_1}^{s_2} \big(\cancel{\mathbf{r} \times \hat{\mathbf{b}}} + \hat{\mathbf{g}} \big) ds$$

$$= \int_{s_1}^{s_2} \Big(\cancel{\mathbf{r} \times \rho_0 \mathbf{r}_{tt}} + \sum_{\alpha=1}^{2} J_\alpha \mathbf{d}_\alpha \times \mathbf{d}_{\alpha,tt} \Big) ds$$

$$\int_{s_1}^{s_2} \big(\mathbf{r}_s \times \mathbf{n} + \cancel{\mathbf{r} \times \mathbf{n}_s} + \mathbf{m}_s \big) ds$$

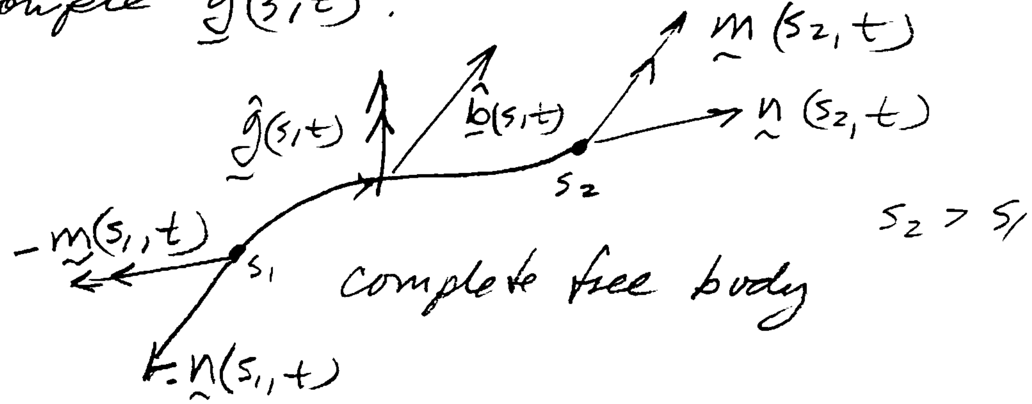
$$\Rightarrow \quad \int_{s_1}^{s_2} \Big(\mathbf{m}_s + \mathbf{r}_s \times \mathbf{n} + \hat{\mathbf{g}} - \sum_{\alpha=1}^{2} J_\alpha \mathbf{d}_\alpha \times \mathbf{d}_{\alpha,tt} \Big) ds = 0 \quad \forall\, (s_1, s_2)$$

$\Rightarrow$ local form:

$$(*) \quad \boxed{\mathbf{m}_s + \mathbf{r}_s \times \mathbf{n} + \hat{\mathbf{g}} = \sum_{\alpha=1}^{2} J_\alpha \mathbf{d}_\alpha \times \mathbf{d}_{\alpha,tt}}$$

<u>Observe</u>: Unlike the 3-d theory, angular momentum balance leads to a system of partial differential equations.

<u>Direct approach</u>: Contact couple $\underset{\sim}{m}(s,t)$ and contact force $\underset{\sim}{n}(s,t)$ acting at (s,t) due to contact with external world, body couple $\hat{\underset{\sim}{g}}(s,t)$:



(LMB) same as p. 126 (couples do not enter the game).

(AMB) $$\underset{\sim}{m}\Big|_{s_1}^{s_2} + (\underset{\sim}{r}\times\underset{\sim}{n})\Big|_{s_1}^{s_2} + \int_{s_1}^{s_2}(\underset{\sim}{r}\times\hat{\underset{\sim}{b}} + \hat{\underset{\sim}{g}})\,ds$$

$$\frac{d}{dt}\int_{s_1}^{s_2}\Big(\underset{\sim}{r}\times\rho_0\underset{\sim}{r}_t + \sum_{\alpha=1}^{2} J_\alpha \underset{\sim}{d}_\alpha\times\underset{\sim}{d}_{\alpha,t}\Big)ds .$$

<u>Exercise</u> (19) A <u>string</u> is a degenerate rod - <u>incapable</u> of <u>sustaining</u> <u>contact</u> <u>couples</u> and <u>body</u> <u>couples</u> and zero cross-sectional inertia $(J_1 = J_2 = 0)$. In this case, show that (AMB) $\Rightarrow$ $\underset{\sim}{n} = T\underset{\sim}{\tau}$, where $\underset{\sim}{\tau} = \underset{\sim}{r}_s/|\underset{\sim}{r}_s|$ is the unit tangent to the curve $s \mapsto \underset{\sim}{r}(s,t)$ and "T" is the tension. (<u>observe</u>: <u>this</u> is similar to the 3-d case ~ an identity)

As yet, we have made no particular assumptions about $\underset{\sim}{d}_1$ & $\underset{\sim}{d}_2$. The most common case is to define $\underset{\sim}{d}_3 \equiv \underset{\sim}{d}_1 \times \underset{\sim}{d}_2$ and assume $\underset{\sim}{d}_i \cdot \underset{\sim}{d}_j = \delta_{ij}$ (orthonormal). This is called the special Cosserat ~~theory~~.

In any case, we have

$$\underset{\sim}{d}_i(s,t) = \underset{\approx}{R}(s,t)\,\underset{\sim}{e}_i \qquad i = 1,2,3$$

for $\underset{\approx}{R}(s,t) \in SO(3)$.

Then (*) $\dot{\underset{\sim}{d}}_i \equiv \underset{\sim}{d}_{i,t} = \dot{\underset{\approx}{R}}\,\underset{\sim}{e}_i = \dot{\underset{\approx}{R}}\,\underset{\approx}{R}^T \underset{\sim}{d}_i$

Now $\underset{\approx}{R}\,\underset{\approx}{R}^T \equiv \underset{\approx}{I} \Rightarrow \dot{\underset{\approx}{R}}\,\underset{\approx}{R}^T + \underset{\approx}{R}\,\dot{\underset{\approx}{R}}^T \equiv \underset{\approx}{0}$

$$\Rightarrow \dot{\underset{\approx}{R}}\,\underset{\approx}{R}^T = -\underset{\approx}{R}\,\dot{\underset{\approx}{R}}^T = -(\dot{\underset{\approx}{R}}\,\underset{\approx}{R}^T)^T$$

or $\dot{\underset{\approx}{R}}\,\underset{\approx}{R}^T \equiv \underset{\approx}{\Omega} \in \text{skew}(\mathbb{E}^3)$.

Then (*) above $\Rightarrow$ $\dot{\underset{\sim}{d}}_i = \underset{\sim}{\omega} \times \underset{\sim}{d}_i$

$$\underset{\sim}{\omega} \equiv \text{axial } \underset{\approx}{\Omega}$$

<u>Fact</u> For <u>any</u> $\underset{\approx}{W} \in \text{skew}(\mathbb{E}^3)$ there is a vector $\underset{\sim}{w}$, called the axial vector $\underset{\approx}{W}$, such ~~that~~

$$\underset{\approx}{W}\,\underset{\sim}{a} = \underset{\sim}{w} \times \underset{\sim}{a} \qquad \forall\ \underset{\sim}{a} \in \mathbb{E}^3.$$

Of course in our case, $\underset{\sim}{\omega}$ is the <u>angular velocity vector</u> of the cross section.

Now $\quad \underset{\sim}{d}_\alpha \times \dot{\underset{\sim}{d}}_\alpha = \underset{\sim}{d}_\alpha \times (\underset{\sim}{\omega} \times \underset{\sim}{d}_\alpha)$

(no sum) $\quad = [\underset{=}{I} - \underset{\sim}{d}_\alpha \otimes \underset{\sim}{d}_\alpha] \underset{\sim}{\omega}$ $\quad$ no sum

$$\Rightarrow \sum_{\alpha=1}^{2} J_\alpha \underset{\sim}{d}_\alpha \times \dot{\underset{\sim}{d}}_\alpha = J_1 (\underset{=}{I} - \underset{\sim}{d}_1 \otimes \underset{\sim}{d}_1) \underset{\sim}{\omega}$$

$$+ J_2 (\underset{=}{I} - \underset{\sim}{d}_2 \otimes \underset{\sim}{d}_2) \underset{\sim}{\omega}$$

call $\underset{=}{J} \equiv (J_1 + J_2) \underset{=}{I} - J_1 \underset{\sim}{d}_1 \otimes \underset{\sim}{d}_1 - J_2 \underset{\sim}{d}_2 \otimes \underset{\sim}{d}_2$

$$\Rightarrow \underset{\text{wrto } \{\underset{\sim}{d}_1, \underset{\sim}{d}_2, \underset{\sim}{d}_3\}}{[J]} = \text{diag}[J_2, J_1, J_1 + J_2]$$

Of course $\underset{=}{J}$ is the mass-moment-of-inertia tensor of the cross section

recall $\begin{cases} J_1 = \displaystyle\int_\Omega X_1^2 \rho_0(\underset{\sim}{X})\, dA \equiv J_{yy} \\ J_2 = J_{xx} \quad \text{about } x\text{-axis} \end{cases}$ conventional notation, about y axis

$J_1 + J_2 = J_{zz}$ "polar" moment of inertia

In any case, the local form of (AMB) can be written

$$\underset{\sim}{m}_s + \underset{\sim}{r}_s \times \underset{\sim}{n} + \underset{\sim}{\hat{q}} = \frac{d}{dt}(\underset{=}{J} \underset{\sim}{\omega}).$$