Constitutive Laws

Recall from pp. 124 & 128:

$$(*)\begin{cases} \underset{\sim}{n} \equiv \int_\Omega \underset{\approx}{S}\,\underset{\sim}{e}_3\, dA\,, & \\ \underset{\sim}{m} \equiv \int_\Omega X_\alpha \underset{\sim}{d}_\alpha \times \underset{\approx}{S}\,\underset{\sim}{e}_3\, dA\,. & = \underset{\sim}{d}_\alpha \times \int_\Omega X_\alpha \underset{\approx}{S}\,\underset{\sim}{e}_3\, dA \end{cases}$$

Again let's looked to the constrained 3-d theory to "see" the appropriate constitutive laws for $\underset{\sim}{n}$ & $\underset{\sim}{m}$, i.e., $\underset{\sim}{n} = \hat{\underset{\sim}{n}}(?)$, $\underset{\sim}{m} = \hat{\underset{\sim}{m}}(?)$:

p. 123 $\quad \underset{\sim}{f}(\underset{\sim}{X}) = \underset{\sim}{r}(s) + X_\alpha \underset{\sim}{d}_\alpha(s)$, (sum $\alpha = 1,2$)

where $\quad \underset{\sim}{X} = X_\alpha \underset{\sim}{e}_\alpha + s\underset{\sim}{e}_3 \qquad (s \equiv X_3)$

or $\quad \underset{\sim}{f}(\underset{\sim}{X}) = \underset{\sim}{r}(\underset{\sim}{e}_3\cdot\underset{\sim}{X}) + (\underset{\sim}{e}_\alpha\cdot\underset{\sim}{X})\underset{\sim}{d}_\alpha(\underset{\sim}{e}_3\cdot\underset{\sim}{X})$

Then $\quad \dfrac{d}{d\alpha}\underset{\sim}{f}(\underset{\sim}{X}+\gamma\underset{\sim}{\eta})\Big|_{\gamma=0} = \underset{\approx}{F}(\underset{\sim}{X})\underset{\sim}{\eta} \qquad \forall\, \underset{\sim}{\eta}$

$$= \frac{d}{d\gamma}\Big\{ \underset{\sim}{r}(\underset{\sim}{e}_3\cdot(\underset{\sim}{X}+\gamma\underset{\sim}{\eta}))$$

$$+ \underset{\sim}{e}_\alpha\cdot(\underset{\sim}{X}+\gamma\underset{\sim}{\eta})\underset{\sim}{d}_\alpha(\underset{\sim}{e}_3\cdot(\underset{\sim}{X}+\gamma\underset{\sim}{\eta}))\Big\}\Big|_{\gamma=0}$$

$$= \underset{\sim}{r}'(s)\,\underset{\sim}{e}_3\cdot\underset{\sim}{\eta} + \underset{\sim}{e}_\alpha\cdot\underset{\sim}{\eta}\,\underset{\sim}{d}_\alpha(s)$$

$$+ \, X_\alpha \, \underset{\sim}{d}_\alpha'(s) \, \underset{\sim}{e}_3 \cdot \underset{\sim}{\eta} \qquad \forall \, \underset{\sim}{\eta}$$

$$\Rightarrow \quad \underset{=}{F}(\underset{-}{X}) = \underset{\sim}{d}_\alpha \otimes \underset{\sim}{e}_\alpha + (\underset{\sim}{r}' + X_\alpha \underset{\sim}{d}_\alpha') \otimes \underset{\sim}{e}_3$$

Now $\quad \underset{=}{S} = \dfrac{\partial W}{\partial \underset{=}{F}}(\underset{=}{F})$

So for the constrained 3-d ~~theory~~, we have

$$\underset{\sim}{n} = \int_\Omega \frac{dW}{d\underset{=}{F}} \left(\underset{\sim}{d}_\alpha \otimes \underset{\sim}{e}_\alpha + (\underset{\sim}{r}_s + X_\alpha \underset{\sim}{d}_{\alpha,s}) \otimes \underset{\sim}{e}_3 \right) \underset{\sim}{e}_3 \, dA$$

$$\equiv \hat{\underset{\sim}{n}}\left(\underset{\sim}{r}_s, \underset{\sim}{d}_\alpha, \underset{\sim}{d}_{\alpha,s}\right)$$

Similarly from (*) p. 133,

$$\underset{\sim}{m} = \hat{\underset{\sim}{m}}\left(\underset{\sim}{r}_s, \underset{\sim}{d}_\alpha, \underset{\sim}{d}_{\alpha,s}\right)$$

We don't interpret this literally, but rather we use it to guide the general form $\big)$.

For material objectivity (MO) p. 41, this general form is OK, and we expect "transforms like a vector"

$$\hat{\underset{\sim}{n}}(\underset{\approx}{Q}\underset{\sim}{r}_s, \underset{\approx}{Q}\underset{\sim}{d}_\alpha, \underset{\approx}{Q}\underset{\sim}{d}_{\alpha,s}) = \underset{\approx}{Q}\,\underset{\sim}{n}(\underset{\sim}{r}_s, \underset{\sim}{d}_\alpha, \underset{\sim}{d}_{\alpha,s})$$

$\hat{\underset{\sim}{m}}$ etc. $\qquad \forall\, \underset{\approx}{Q} \in SO(3)$

This is the approach taken in Antman's book. However, for material symmetry and hyperelasticity, this form is woefully lacking.

So instead we consider the constrained free energy

$$W(\underset{\approx}{F}) = W\left(\underset{\sim}{d}_\alpha \otimes \underset{\sim}{e}_\alpha + (\underset{\sim}{r}_s + X_\alpha \underset{\sim}{d}_{\alpha,s}) \otimes \underset{\sim}{e}_3\right).$$

We do two things:

① recall $\underset{\sim}{d}_i = \underset{\approx}{R}\,\underset{\sim}{e}_i$ ← rotation of the cross section

② integrate $W(\underset{\approx}{F})$ over Ω

Define

$$\Phi(\underset{\sim}{r}_s, \underset{\approx}{R}, \underset{\approx}{R}_s) \leftarrow \left(\text{Free energy density per unit undef. ~~length~~}\right)$$

$$\equiv \int_\Omega W\left(\underset{\approx}{R}\,\underset{\sim}{e}_\alpha \otimes \underset{\sim}{e}_\alpha + (\underset{\sim}{r}_s + X_\alpha \underset{\approx}{R}_s \underset{\sim}{e}_\alpha) \otimes \underset{\sim}{e}_3\right) dA$$

Again, we do not interpret this literally, but rather use it as a guide for the form of the left side.

Material Objectivity

$$F \to Q F, \quad F \equiv R\, e_\alpha \otimes e_\alpha + (r_s + X_\alpha R_s e_\alpha) \otimes e_3$$

$$\therefore Q F = Q R\, e_\alpha \otimes e_\alpha + (Q r_s + X_\alpha Q R_s e_\alpha) \otimes e_3$$

$$\Rightarrow r_s \to Q r_s, \quad R \to Q R, \quad R_s \to Q R_s .$$

<u>Defn</u> (MO) $\quad \Phi(Q r_s, Q R, Q R_s) = \Phi(r_s, R, R_s)$
$\forall\, Q \in SO(3)$.

We exploit (MO) precisely as before (p. 41): choose $Q \equiv R^T \Rightarrow$

$$\Phi(R^T r_s, \overset{I}{\cancel{R^T R}}, R^T R_s) = \Phi(r_s, R, R_s)$$

In other words,

$$(*) \quad \Phi(r_s, R, R_s) = \Psi(R^T r_s, R^T R_s)$$

To proceed further, we consider the spatial version of the calculations on p. 131:

$$\tfrac{d}{ds}(\,) \quad d_i' = R' e_i$$
$$= R' R^T d_i \qquad R' R^T \equiv K \text{ skew}$$

Note that we may express (*) as

$$(*) \quad \Phi(\mathbf{r}', \mathbf{R}, \mathbf{R}') = \Psi(\mathbf{R}^T\mathbf{r}', \mathbf{R}^T\mathbf{R}'\overbrace{\mathbf{R}^T\mathbf{R}}^{\mathbf{I}})$$

$$= \Psi(\mathbf{R}^T\mathbf{r}', \mathbf{R}^T\mathbf{K}\mathbf{R})$$

Define $\quad \mathbf{r}' = \nu_i \mathbf{d}_i \quad , \nu_i = \mathbf{d}_i \cdot \mathbf{r}_s$

$$\mathbf{K} = K_{ij}\, \mathbf{d}_i \otimes \mathbf{d}_j \,, \quad K_{ij} = \mathbf{d}_i \cdot \mathbf{K}\mathbf{d}_j$$

Since $\mathbf{K} \in \text{skew}(\mathbb{E}^3)$, we have

$$[K_{ij}] = \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix}$$

Then $\quad \mathbf{K}\mathbf{a} = \mathbf{k} \times \mathbf{a} \quad$, where $\mathbf{k} = k_i \mathbf{d}_i$.

Now

$$\mathbf{R}^T\mathbf{r}_s = \nu_i \mathbf{R}^T\mathbf{d}_i = \nu_i \mathbf{e}_i$$

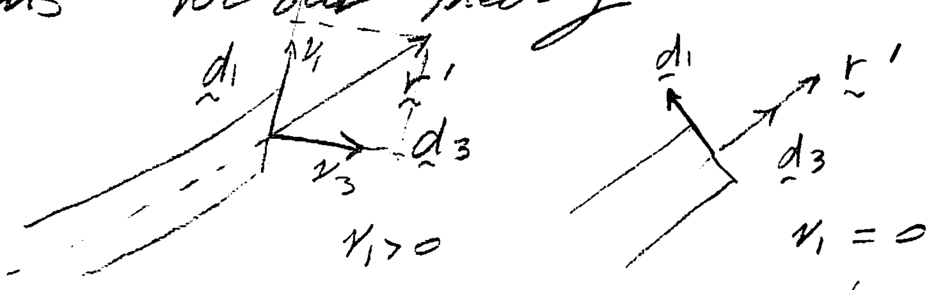
$$\mathbf{R}^T\mathbf{K}\mathbf{R} = K_{ij}\, \mathbf{R}^T\mathbf{d}_i \otimes \mathbf{R}^T\mathbf{d}_j = K_{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j$$

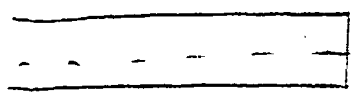
So from (*) above, we ~~see that~~ (MO) $\Leftrightarrow$

$$\boxed{\Phi(\mathbf{r}', \mathbf{R}, \mathbf{R}') = \Upsilon(\nu_1, \nu_2, \nu_3, k_1, k_2, k_3)}$$

$$= \tilde{\Upsilon}(\mathbf{R}^T\mathbf{r}', \mathbf{R}^T\mathbf{k})$$

The scalar quantities ν_i's & κ_i's are the "strains" for our theory:





ν_1 & ν_2 ≈ shears

ν_3 ≈ stretch

Note: If $\nu_1 = \nu_2 \equiv 0$ (prescribed), the rod is said to be unshearable, in which case $\nu_3 = |\underset{\sim}{r}'|$, which is precisely the stretch of the centerline. If $\nu_3 \equiv 1$ (prescribed), then the rod is said to be **inextensible**.

In analogy with dynamics & angular velocity (p. 131), note that the κ_i represent "infinitesimal" rotations about the $\underset{\sim}{d}_i$ axis as we move spatially along the centerline. Accordingly

κ_1, κ_2 ≈ bending curvatures

κ_3 ≈ twist

For unshearable, inextensible rods κ_1 & κ_2 are precisely curvatures (as defined in differential geometry).

Local injectivity: $\det \underline{F} > 0$

(pp. 134, 136) $\underline{F} = \underset{\approx}{R}\left(\underset{\sim}{e}_\alpha \otimes \underset{\sim}{e}_\alpha + \underset{\approx}{R}^T \underset{\sim}{r}' \otimes \underset{\sim}{e}_3\right.$

$\left. + \cancel{X_\alpha \underset{\approx}{R}^T \underset{\approx}{R}' \underset{\sim}{e}_\alpha \otimes \underset{\sim}{e}_3}\right)$ usually ignored

$= \underset{=}{R}\left(\underset{\sim}{e}_\alpha \otimes \underset{\sim}{e}_\alpha + \nu_i \underset{\sim}{e}_i \otimes \underset{\sim}{e}_3\right)$

$\det \underline{F} = \cancel{\det \underline{R}}^{\,1} \det\left(\underset{\sim}{e}_\alpha \otimes \underset{\sim}{e}_\alpha + \nu_i \underset{\sim}{e}_i \otimes \underset{\sim}{e}_3\right)$

$= \det \begin{bmatrix} 1 & 0 & \nu_1 \\ 0 & 1 & \nu_2 \\ 0 & 0 & \nu_3 \end{bmatrix}$

$= \nu_3 = \underset{\sim}{r}' \cdot \underset{\sim}{d}_3 > 0$

More on "strains": simple examples

① $\underset{\sim}{r}(s) = \lambda s \underset{\sim}{e}_3 \quad \underset{\sim}{d}_1 \equiv \underset{\sim}{e}_1,\ \underset{\sim}{d}_2 \equiv \underset{\sim}{e}_2,\ \underset{\sim}{d}_3 \equiv \underset{\sim}{e}_3$

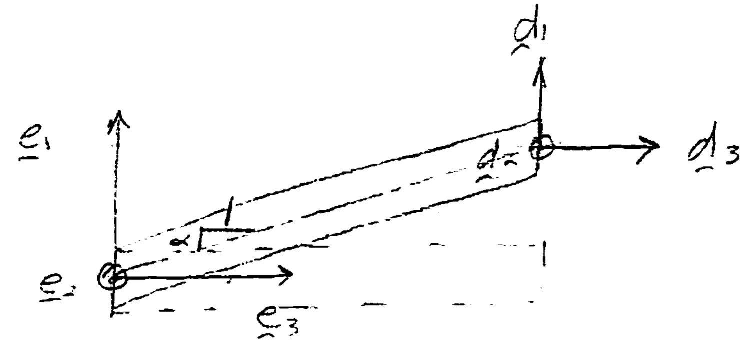
Then $\left.\begin{aligned} \underset{\sim}{r}' &= \lambda \underset{\sim}{e}_3 \\ &= \lambda \underset{\sim}{d}_3 \end{aligned}\right\} \Rightarrow \nu_3 = \lambda$ pure stretch

Note: In general, $\nu_3 = \underset{\sim}{d}_3 \cdot \underset{\sim}{r}' = \underset{\sim}{r}' \cdot (\underset{\sim}{d}_1 \times \underset{\sim}{d}_2)$
volume of a parallelepiped

Antman calls ν_3 "the dilatation". Also, in

general, the true stretch is given by
$|\mathbf{r}'| = \sqrt{\nu_1^2+\nu_2^2+\nu_3^2}$.

(2) $\mathbf{r}(s) = s(\alpha\mathbf{e}_1 + \mathbf{e}_3)$, $\mathbf{d}_1 \equiv \mathbf{e}_1$, $\mathbf{d}_2 \equiv \mathbf{e}_2$, $\mathbf{d}_3 \equiv \mathbf{e}_3$



$\therefore\ \mathbf{r}' = \alpha\mathbf{e}_1 + \mathbf{e}_3 = \alpha\mathbf{d}_1 + \mathbf{d}_3$

$\Rightarrow\ \nu_1 = \alpha,\ \nu_2 = 0,\ \nu_3 = 1$

"shear"

(3) $\mathbf{r}(s) = s\mathbf{e}_3$, $\mathbf{d}_1(s) = \cos(ws)\,\mathbf{e}_1 + \sin(ws)\,\mathbf{e}_2$

$\mathbf{d}_2(s) = \mathbf{e}_3 \times \mathbf{d}_1(s)$

$\mathbf{d}_3(s) = \mathbf{e}_3$

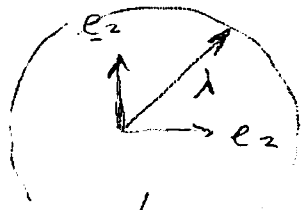
$\mathbf{d}_1' = w(-\sin(ws)\,\mathbf{e}_1 + \cos(ws)\,\mathbf{e}_2) = w\,\mathbf{e}_3 \times \mathbf{d}_1(s)$

$\mathbf{d}_2' = w(-\cos(ws)\,\mathbf{e}_1 - \sin(ws)\,\mathbf{e}_2) = w\,\mathbf{e}_3 \times \mathbf{d}_2(s)$

$\Rightarrow\ \boldsymbol{\kappa} = \omega\,\mathbf{e}_3 = \omega\,\mathbf{d}_3 \Rightarrow \kappa_3 = \omega,\ (\kappa_1 = \kappa_2 = 0)$

$\omega \sim$ twist

(4) $\mathbf{r}(s) = \lambda(\cos(\gamma s)\,\mathbf{e}_2 + \sin(\gamma s)\,\mathbf{e}_3)$



$\mathbf{d}_1 \equiv \mathbf{e}_1$ $\quad \mathbf{d}_2 = \cos(\gamma s)\,\mathbf{e}_2 + \sin(\gamma s)\,\mathbf{e}_3$ $\quad \mathbf{d}_3 = \mathbf{e}_1 \times \mathbf{d}_2$

$$\underset{\sim}{r}' = \lambda\gamma'(-\sin(\gamma s)\,\underset{\sim}{e}_2 + \cos(\gamma s)\,\underset{\sim}{e}_3)$$

$$= \lambda\gamma'\,\underset{\sim}{d}_3 \quad \Rightarrow \quad \nu_3 = \lambda\gamma'$$

$$\underset{\sim}{d}_2' = \gamma'\underset{\sim}{d}_3 = \gamma'\underset{\sim}{e}_1 \times \underset{\sim}{d}_2 = (\gamma'\underset{\sim}{d}_1) \times \underset{\sim}{d}_2$$

$$\underset{\sim}{d}_3' = -\gamma'\underset{\sim}{d}_2 = \gamma'\underset{\sim}{e}_1 \times \underset{\sim}{d}_3 = (\gamma'\underset{\sim}{d}_1) \times \underset{\sim}{d}_3$$

$$\Rightarrow \underset{\uparrow}{\kappa_1 = \gamma'} \quad (\kappa_2 = \kappa_3 = 0)$$

"flexural strain"

(note that $\frac{1}{\lambda}$ is the curvature of the circle).

$\Rightarrow$ thus, κ_1 is not influenced by changes in curvature due to inflation

Lecture 17

Constitutive Laws for Force & Moment

* (Claim $n_i = \dfrac{\partial \Psi}{\partial \nu_i}$, $m_i = \dfrac{\partial \Psi}{\partial \kappa_i}$, $i = 1, 2, 3.$)

One way to see this is to return to the constrained 3-d case (p. 135):

$$\underset{=}{F} = \underset{=}{R}\left[\underset{\sim}{e}_\alpha \otimes \underset{\sim}{e}_\alpha + (\underset{=}{R}^T \underset{\sim}{r}' + X_\alpha \underset{=}{R}^T \underset{=}{R}' \underset{=}{R}^T \underset{=}{R}\,\underset{\sim}{e}_\alpha) \otimes \underset{\sim}{e}_3\right]$$

$$= \underset{=}{R}\left[\underset{\sim}{e}_\alpha \otimes \underset{\sim}{e}_\alpha + (\underset{=}{R}^T \underset{\sim}{r}' + X_\alpha \underset{=}{R}^T \underset{=}{K}\,\underset{=}{R}\,\underset{\sim}{e}_\alpha) \otimes \underset{\sim}{e}_3\right]$$

$$= \underset{=}{R}\left[\underset{\sim}{e}_\alpha \otimes \underset{\sim}{e}_\alpha + (\nu_i \underset{\sim}{e}_i + X_\alpha \kappa_i \underset{\sim}{e}_i \times \underset{\sim}{e}_\alpha) \otimes \underset{\sim}{e}_3\right]$$