$$\underset{\sim}{r}' = \lambda\gamma(-\sin(\gamma s)\underset{\sim}{e}_2 + \cos(\gamma s)\underset{\sim}{e}_3)$$

$$= \lambda\gamma\,\underset{\sim}{d}_3 \quad \Rightarrow \quad \nu_3 = \lambda\gamma$$

$$\underset{\sim}{d}_2' = \gamma\underset{\sim}{d}_3 = \gamma\underset{\sim}{e}_1 \times \underset{\sim}{d}_2 = (\gamma\underset{\sim}{d}_1) \times \underset{\sim}{d}_2$$

$$\underset{\sim}{d}_3' = -\gamma\underset{\sim}{d}_2 = \gamma\underset{\sim}{e}_1 \times \underset{\sim}{d}_3 = (\gamma\underset{\sim}{d}_1) \times \underset{\sim}{d}_3$$

$$\Rightarrow \kappa_1 = \gamma \quad (\kappa_2 = \kappa_3 = 0)$$

"flexural strain"

(note that $\frac{1}{\lambda}$ is the curvature of the circle).

$\Rightarrow$ thus, κ_1 is not influenced by changes in curvature due to inflation

Lecture 17

Constitutive Laws for Force & Moment

$*$ (Claim $\quad n_i = \dfrac{\partial \Upsilon}{\partial \nu_i}, \quad m_i = \dfrac{\partial \Upsilon}{\partial \kappa_i}, \quad i = 1, 2, 3.$)

One way to see this is to return to the constrained 3-d case (p. 135):

$$\underset{=}{F} = \underset{=}{R}\left[\underset{\sim}{e}_\alpha \otimes \underset{\sim}{e}_\alpha + (\underset{=}{R}^T\underset{\sim}{r}' + X_\alpha \underset{=}{R}^T\underset{=}{R}'\underset{=}{R}^T\underset{=}{R}\,\underset{\sim}{e}_\alpha) \otimes \underset{\sim}{e}_3\right]$$

$$= \underset{=}{R}\left[\underset{\sim}{e}_\alpha \otimes \underset{\sim}{e}_\alpha + (\underset{=}{R}^T\underset{\sim}{r}' + X_\alpha \underset{=}{R}^T\underset{=}{K}\,\underset{=}{R}\,\underset{\sim}{e}_\alpha) \otimes \underset{\sim}{e}_3\right]$$

$$= \underset{=}{R}\left[\underset{\sim}{e}_\alpha \otimes \underset{\sim}{e}_\alpha + (\nu_i \underset{\sim}{e}_i + X_\alpha \kappa_i \underset{\sim}{e}_i \times \underset{\sim}{e}_\alpha) \otimes \underset{\sim}{e}_3\right]$$

where $\kappa_i \underset{\sim}{e}_i = \mathrm{axial}(\underline{\underline{R}}^T \underline{\underline{K}}\, \underline{\underline{R}}) = \mathrm{axial}(\kappa_{ij} \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j)$.

From pp. 135 & 137, we have

$$\Upsilon(\nu_1, \nu_2, \nu_3, \kappa_1, \kappa_2, \kappa_3)$$

$$= \int_\Omega W\left(\underline{\underline{R}}\left[\underset{\sim}{e}_\alpha \otimes \underset{\sim}{e}_\alpha + (\nu_i \underset{\sim}{e}_i + X_\alpha \kappa_i \underset{\sim}{e}_i \times \underset{\sim}{e}_\alpha) \otimes \underset{\sim}{e}_3\right]\right) dA$$

$$\therefore \frac{\partial \Upsilon}{\partial \nu_i} = \int_\Omega \underbrace{\frac{\partial W}{\partial \underline{\underline{F}}}(\quad)}_{\underline{\underline{S}}} \cdot \overbrace{\underline{\underline{R}}\underset{\sim}{e}_i}^{d_i} \otimes \underset{\sim}{e}_3 \, dA$$

$$= \underset{\sim}{d}_i \cdot \underbrace{\int_\Omega \underline{\underline{S}}\, \underset{\sim}{e}_3 \, dA}_{\underset{\sim}{n}\ (p.\ 124)} = n_i$$

$$\frac{\partial \Upsilon}{\partial \kappa_i} = \int_\Omega \underbrace{\frac{dW}{d\underline{\underline{F}}}(\,)}_{\underline{\underline{S}}} \cdot X_\alpha \underline{\underline{R}}(\underset{\sim}{e}_i \times e_\alpha) \otimes \underset{\sim}{e}_3$$

$$= \int_\Omega X_\alpha \underbrace{\underline{\underline{R}}(\underset{\sim}{e}_i \times \underset{\sim}{e}_\alpha)}_{\text{Lemma p. 27}} \cdot \underline{\underline{S}}\, \underset{\sim}{e}_3 \, dA$$

$$\underline{\underline{R}}(\underset{\sim}{e}_i \times \underset{\sim}{e}_\alpha) = \det \underline{\underline{R}}\, \underline{\underline{R}}^{-T}(\underset{\sim}{e}_i \times \underset{\sim}{e}_\alpha) = \underline{\underline{R}}\underset{\sim}{e}_i \times \underline{\underline{R}}\underset{\sim}{e}_\alpha$$

$$= \int_\Omega X_\alpha (\underset{\sim}{d}_i \times \underset{\sim}{d}_\alpha) \cdot \underline{\underline{S}}\, \underset{\sim}{e}_3 \, dA$$

(scalar triple product)

$$= \underset{\sim}{d}_i \cdot \underbrace{\left(\underset{\sim}{d}_\alpha \times \int_\Omega X_\alpha \underline{\underline{S}}\, \underset{\sim}{e}_3 \, dA\right)}_{\underset{\sim}{m}\ (p.\ 128)} \equiv m_i$$

We can also establish (*) p. 141 "directly" via stationary potential energy. In particular, we wish to immitate the approach of (SPE) from p. 83 – but now solely in the context of the special Cosserat theory of rods.

$$\mathcal{V}[\underset{\sim}{r}, \underset{=}{R}] = \int_0^L \Psi\left(\underset{=}{R}^T \underset{\sim}{r}', \underset{=}{R}^T \underset{\sim}{k}\right) ds$$

configuration variables

p. 137 $\underset{=}{K}\underset{\sim}{a} = \underset{\sim}{k} \times \underset{\sim}{a} \quad \forall \underset{\sim}{a}$

<u>Admissibility</u>: An <u>admissible variation</u> for $\underset{\sim}{r}$ is $\frac{d}{d\alpha}\left(\underset{\sim}{r} + \alpha \underset{\sim}{\eta}\right)\Big|_{\alpha=0} = \underset{\sim}{\eta}$ suff. smooth + b.c.'s (not specified)

However, life is much more complex for $\underset{=}{R}$: We can not write $\frac{d}{d\alpha}\left(\underset{=}{R} + \alpha \underset{=}{A}\right)\Big|_{\alpha=0} = \underset{=}{A}$ – because $\underset{=}{R} + \alpha \underset{=}{A}$ is <u>not</u> generally a rotation (is not generally in $SO(3)$) for small α and $\underset{=}{A}$ in $L(\mathbb{E}^3)$ (or even $\underset{=}{A} \in SO(3)$!)

Instead, we must perturb $\underset{=}{R}$ in a way the preserves membership in $SO(3)$:

← suff. smooth + b.c. (unspecified)

Let $\underline{\underline{\Theta}}(s) \in \mathrm{Skew}(\mathbb{E}^3)$ denote a field. We perturb $\underline{\underline{R}}(s)$ via

$$\exp(\alpha \underline{\underline{\Theta}})\underline{\underline{R}} \quad \text{and} \quad \text{compute}$$

$$\frac{d}{d\alpha} \exp(\alpha \underline{\underline{\Theta}})\underline{\underline{R}} \Big|_{\alpha=0} \overset{(p.44)}{=} \underline{\underline{\Theta}} \exp(\alpha \underline{\underline{\Theta}}) \underline{\underline{R}} \Big|_{\alpha=0}$$

$$= \underline{\underline{\Theta}} \cancelto{\underline{\underline{I}}}{\exp(\underline{\underline{0}})} \underline{\underline{R}} = \underline{\underline{\Theta}}\,\underline{\underline{R}}$$

This takes care of $\underline{\underline{R}}$. What about $\underset{\sim}{K}$?
Well,

$$\underline{\underline{R}} \to \exp(\alpha \underline{\underline{\Theta}})\underline{\underline{R}}$$

$$\underline{\underline{R}}' = \underbrace{\frac{d}{ds}\left(\exp(\alpha \underline{\underline{\Theta}})\right)}_{?} \underline{\underline{R}} + \exp(\alpha \underline{\underline{\Theta}}) \underline{\underline{R}}'$$

For this we need the power-series definition:

$$\exp(\underline{\underline{\Theta}}) = \underline{\underline{I}} + \underline{\underline{\Theta}} + \frac{1}{2} \underline{\underline{\Theta}}\,\underline{\underline{\Theta}} + \frac{1}{3!} \underline{\underline{\Theta}}\,\underline{\underline{\Theta}}\,\underline{\underline{\Theta}} + \cdots$$

$$\exp(\alpha \underline{\underline{\Theta}}) = \underline{\underline{I}} + \alpha \underline{\underline{\Theta}} + \frac{\alpha^2}{2} \underline{\underline{\Theta}}\,\underline{\underline{\Theta}} + \frac{\alpha^3}{3!} \underline{\underline{\Theta}}\,\underline{\underline{\Theta}}\,\underline{\underline{\Theta}} + \cdots$$

$$\frac{d}{ds} \exp(\alpha \underline{\underline{\Theta}}) = \alpha \underline{\underline{\Theta}}' + \frac{\alpha^2}{2}\left(\underline{\underline{\Theta}}'\underline{\underline{\Theta}} + \underline{\underline{\Theta}}\,\underline{\underline{\Theta}}'\right) + \cdots$$

$$\underline{\underline{K}} = \underline{\underline{R}}'\underline{\underline{R}}^T$$

$$\underline{\underline{R}}^T \to \underline{\underline{R}}^T \left(\exp(\alpha \underline{\underline{\Theta}})\right)^T$$

$$\left(\exp(\alpha \underset{=}{\Theta})\right)^T = \overbrace{\underset{=}{I} + \alpha \underset{=}{\Theta}^T + \frac{\alpha^2}{2} \underset{=}{\Theta}^T \underset{=}{\Theta}^T +}^{\exp(\alpha \underset{=}{\Theta}^T)}$$

$$= \underset{=}{I} - \alpha \underset{=}{\Theta} + \frac{\alpha^2}{2} \underset{=}{\Theta}\,\underset{=}{\Theta} - \cdots$$

$$= \exp(-\alpha \underset{=}{\Theta})$$

$$\therefore \underset{=}{K} \rightarrow \left\{ \left[(\alpha \underset{=}{\Theta}' + \cdots) \underset{=}{R} + \exp(\alpha \underset{=}{\Theta}) \underset{=}{R}' \right] \underset{=}{R}^T \exp(\alpha \underset{=}{\Theta}^T) \right\}$$

$$\frac{d}{d\alpha} \{ \quad \} \Big|_{\alpha = 0}$$

$$= \underset{=}{\Theta}' + \underset{=}{\Theta}\,\underset{=}{K} - \underset{=}{K}\,\underset{=}{\Theta}$$

Exercise 20 Show that $\underset{\approx}{\Theta} \underset{=}{K} - \underset{=}{K} \underset{=}{\Theta} \in$ Skew$(\mathbb{E}^3)$ and that

$$(\underset{=}{\Theta} \underset{=}{K} - \underset{=}{K} \underset{=}{\Theta}) \underset{\sim}{h} = (\underset{\sim}{\theta} \times \underset{\sim}{\kappa}) \times \underset{\sim}{h} \quad \forall \underset{\sim}{h} \in \mathbb{E}^3,$$

where $\underset{\sim}{\theta} = \text{axial}\ \underset{=}{\Theta}$ and $\underset{\sim}{\kappa} = \text{axial}\ \underset{=}{K}$.

Hint: $\underset{\sim}{a} \times (\underset{\sim}{b} \times \underset{\sim}{c}) = (\underset{\sim}{a} \cdot \underset{\sim}{c}) \underset{\sim}{b} - (\underset{\sim}{a} \cdot \underset{\sim}{b}) \underset{\sim}{c}$

According to Exer. 20,

$$\text{axial}\left(\underset{=}{\Theta}' + \underset{=}{\Theta} \underset{=}{K} - \underset{=}{K} \underset{=}{\Theta} \right) = \underset{\sim}{\theta}' + \underset{\sim}{\theta} \times \underset{\sim}{\kappa}$$

$$\rightarrow \underset{\approx}{K} \rightarrow \underset{=}{K} + \alpha \left(\underset{=}{\Theta}' + \underset{=}{\Theta} \underset{=}{K} - \underset{=}{K} \underset{=}{\Theta} \right) + o(\alpha)$$

$$\Rightarrow \underset{\sim}{\kappa} \longrightarrow \underset{\sim}{\kappa} + \alpha \left(\underset{\sim}{\theta}' + \underset{\sim}{\theta} \times \underset{\sim}{\kappa} \right) + o(\alpha)$$