Lecture 18

$$\therefore \quad \mathbf{R}^T\mathbf{r}' \to \mathbf{R}^T \exp(-\alpha\boldsymbol{\Theta})(\mathbf{r}' + \alpha\boldsymbol{\eta}')$$

$$\frac{d}{d\alpha}\, \mathbf{R}^T \exp(-\alpha\boldsymbol{\Theta})(\mathbf{r}' + \alpha\boldsymbol{\eta}')\Big|_{\alpha=0}$$

$$= \mathbf{R}^T(-\boldsymbol{\Theta}\mathbf{r}' + \boldsymbol{\eta}') = \mathbf{R}^T(-\boldsymbol{\theta}\times\mathbf{r}' + \boldsymbol{\eta}')$$

$$\therefore \quad \mathbf{R}^T\mathbf{r}' \to \mathbf{R}^T\mathbf{r}' + \alpha\,\mathbf{R}^T(\boldsymbol{\eta}' - \boldsymbol{\theta}\times\mathbf{r}') + o(\alpha)$$

$$\mathbf{R}^T\mathbf{k} \to \underbrace{\mathbf{R}^T \exp(-\alpha\boldsymbol{\Theta})\left[\mathbf{k} + \alpha(\boldsymbol{\theta}' + \boldsymbol{\theta}\times\mathbf{k}) + \cdots\right]}$$

$$\frac{d}{d\alpha}\left\{ \quad \right\}\Big|_{\alpha=0} = \mathbf{R}^T\left\{-\boldsymbol{\Theta}\mathbf{k} + \boldsymbol{\theta}' + \boldsymbol{\theta}\times\mathbf{k}\right\}$$

$$= \mathbf{R}^T\left\{-\cancel{\boldsymbol{\theta}\times\mathbf{k}} + \boldsymbol{\theta}' + \cancel{\boldsymbol{\theta}\times\mathbf{k}}\right\}$$

$$\Rightarrow \quad \mathbf{R}^T\mathbf{k} \to \mathbf{R}^T\mathbf{k} + \alpha\,\mathbf{R}^T\boldsymbol{\theta}' + o(\alpha)$$

$$\therefore \quad \delta V = \frac{d}{d\alpha} V\left[\mathbf{r} + \alpha\boldsymbol{\eta}, \exp(\alpha\boldsymbol{\Theta})\mathbf{R}\right]\Big|_{\alpha=0}$$

$$= \frac{d}{d\alpha}\int_0^{L \,\leftarrow\, \text{length of rod}} \Psi\left(\mathbf{R}^T[\mathbf{r}' + \alpha(\boldsymbol{\eta}' - \boldsymbol{\theta}\times\mathbf{r}')] + \cdots, \mathbf{R}^T(\mathbf{k} + \alpha\boldsymbol{\theta}') + \cdots\right) - \delta W_{ext} = 0 \quad \Big|_{\alpha=0}$$

$$= \int_0^L \Big[\frac{\partial\Psi}{\partial\mathbf{v}}(\mathbf{R}^T\mathbf{r}', \mathbf{R}^T\mathbf{k})\cdot\mathbf{R}^T(\boldsymbol{\eta}' - \boldsymbol{\theta}\times\mathbf{r}')$$

$$+ \frac{\partial\Psi}{\partial\mathbf{k}}(\mathbf{R}^T\mathbf{r}', \mathbf{R}^T\mathbf{k})\cdot\mathbf{R}^T\boldsymbol{\theta}'\Big]\, ds - \delta W_{ext} = 0 \qquad \forall \text{ adm } \boldsymbol{\eta}, \boldsymbol{\theta},$$

where $\dfrac{\partial \Psi}{\partial \underset{\sim}{\nu}} \equiv \dfrac{\partial \Psi}{\partial \nu_i} \underset{\sim}{e}_i$, $\dfrac{\partial \Psi}{\partial \underset{\sim}{K}} \equiv \dfrac{\partial \Psi}{\partial K_i} \underset{\sim}{e}_i$.

$$\therefore \delta V = \int_0^L \Big[\frac{\partial \Psi}{\partial \nu_i} \underset{\sim}{d}_i \cdot (\underset{\sim}{\eta}' - \underset{\sim}{\theta} \times \underset{\sim}{r}') \, ds$$

$$+ \frac{\partial \Psi}{\partial K_i} \underset{\sim}{d}_i \cdot \underset{\sim}{\theta}' \Big] ds - \delta W_{ext} = 0$$

$\forall$ adm $\underset{\sim}{\eta}, \underset{\sim}{\theta}$

Integrate by parts:

$$\delta V = \int_0^L \Big[-\left(\frac{\partial \Psi}{\partial \nu_i} \underset{\sim}{d}_i\right)' \cdot \underset{\sim}{\eta} - \left(\frac{\partial \Psi}{\partial K_i} \underset{\sim}{d}_i\right)' \cdot \underset{\sim}{\theta}$$

(triple prod)

$$- \underset{\sim}{\theta} \cdot \underset{\sim}{r}' \times \left(\frac{\partial \Psi}{\partial \nu_i} \underset{\sim}{d}_i\right)$$

$$- \underbrace{(\hat{\underset{\sim}{b}} \cdot \underset{\sim}{\eta} + \hat{\underset{\sim}{g}} \cdot \underset{\sim}{\theta})}_{\delta W_{ext}} \Big] ds + \text{boundary terms} = 0 \quad \forall \text{ adm } \underset{\sim}{\eta}, \underset{\sim}{\theta}$$

$$\Rightarrow \quad \left(\frac{\partial \Psi}{\partial \nu_i} \underset{\sim}{d}_i\right)' + \hat{\underset{\sim}{b}} = \underset{\sim}{0}$$

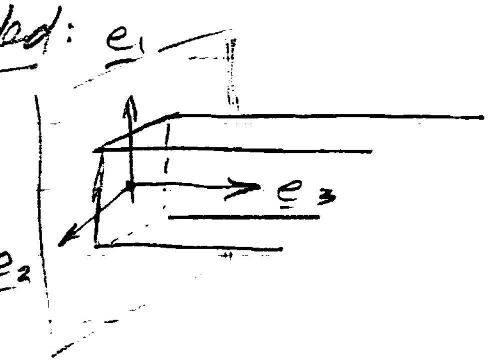
$$\left(\frac{\partial \Psi}{\partial K_i} \underset{\sim}{d}_i\right)' + \underset{\sim}{r}' \times \left(\frac{\partial \Psi}{\partial \nu_i} \underset{\sim}{d}_i\right) + \hat{\underset{\sim}{g}} = \underset{\sim}{0}$$

Compare pp. 126, 129

$$\left.\begin{aligned} &\Rightarrow \quad \underset{\sim}{n} = \frac{\partial \Psi}{\partial \nu_i} \underset{\sim}{d}_i \Rightarrow \boxed{n_i = \frac{\partial \Psi}{\partial \nu_i}} \\ &\quad\;\; \underset{\sim}{m} = \frac{\partial \Psi}{\partial K_i} \underset{\sim}{d}_i \Rightarrow \boxed{m_i = \frac{\partial \Psi}{\partial K_i}} \end{aligned}\right\} \quad i = 1, 2, 3 .$$

Admissibility for Geometric Boundary Conditions (Typical)

① Welded:



$$\underset{\sim}{r}(0) = \underset{\sim}{0}, \quad \underset{=}{R}(0) = \underset{=}{I}$$

For the former $\quad \overset{\underset{\sim}{0}}{\cancel{\underset{\sim}{r}(0)}} + \alpha\, \underset{\sim}{\eta}(0) = \underset{\sim}{0}$

$$\Rightarrow \boxed{\underset{\sim}{\eta}(0) = \underset{\sim}{0}} \quad \text{for admissibility}$$

What about $\underset{\sim}{\theta}(0)$?

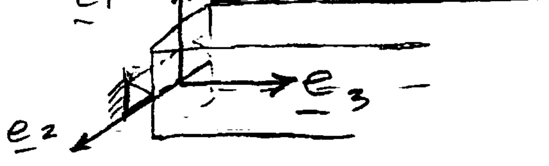
Note: $\quad \exp(\alpha\, \underset{\approx}{\Theta}(0))\, \overset{\underset{=}{I}}{\cancel{R(0)}} = \underset{=}{I} \quad (\forall \alpha)$

Then $\quad \left.\dfrac{d}{d\alpha}\left(\exp(\alpha\, \underset{=}{\Theta}(0)) = \underset{=}{I}\right)\right|_{\alpha=0}$

$$\Rightarrow \underset{=}{\Theta}(0) = \underset{=}{0}$$

$$\Rightarrow \boxed{\underset{\sim}{\theta}(0) = \underset{\sim}{0}} \quad \text{for admissibility.}$$

② Pinned (hinged):



$$\underset{\sim}{r}(0) = \underset{\sim}{0}$$

$$\underset{\approx}{R}(0)\, \underset{\sim}{e}_2 = \underset{\sim}{e}_2$$

Clearly $\underset{\sim}{\eta}(0) = \underset{\sim}{0}$ (as in ①)

Now $$\exp(\alpha \underline{\underline{\Theta}}(0)) \overbrace{\underline{\underline{R}}(0)\, \underset{\sim}{e}_z}^{\underset{\sim}{e}_z} \equiv \underset{\sim}{e}_z \quad (\forall \alpha)$$

$$\Rightarrow \frac{d}{d\alpha}\left(\exp(\alpha \underline{\underline{\Theta}}(0))\, \underset{\sim}{e}_z = \underset{\sim}{e}_z \right)\Big|_{\alpha=0}$$

$$\Rightarrow \underline{\underline{\Theta}}(0)\, \underset{\sim}{e}_z = \underset{\sim}{0}$$

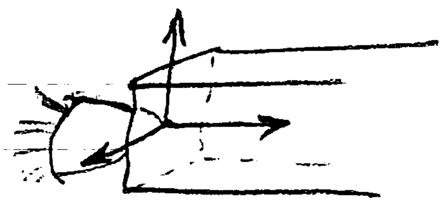
$$\text{or} \quad \underset{\sim}{\theta}(0) \times \underset{\sim}{e}_z = \underset{\sim}{0}$$

$$\Rightarrow \boxed{\theta_1(0) = \theta_3(0) = 0} \quad (\underset{\sim}{\theta}(0) \parallel \text{to } \underset{\sim}{e}_z)$$

What about another b.c.? Yes - but it is a natural b.c. coming from the integration by parts

Exercise (21) In case ② above show that the vanishing of the 1st variation (p. 147) $\Rightarrow$ $\underset{\sim}{m}(0) \cdot \underset{\sim}{e}_z = 0$.

③ Ball-in-Socket:



$$\underset{\sim}{r}(0) = 0$$

$$\Rightarrow \underset{\sim}{\eta}(0) = \underset{\sim}{0}$$

for admissibility

Let's derive the natural boundary conditions in this case. Observe that there are no restrictions on $\underset{\sim}{\theta}(0)$. Going back to the weak form at the top of p. 147, we have the term

$$\cdots \int_0^L \overbrace{\frac{\partial W}{\partial \kappa_i} \underset{\sim}{d_i}}^{\underset{\sim}{m}} \cdot \underset{\sim}{\theta}' \, ds$$

integrate by parts:

$$= \int_0^L -\underset{\sim}{m}' \cdot \underset{\sim}{\theta} + \underset{\sim}{m} \cdot \underset{\sim}{\theta} \Big|_0^L$$

Since $\underset{\sim}{\theta}(0)$ is arbitrary, we conclude ~~that~~

$\boxed{\underset{\sim}{m}(0) = \underset{\sim}{0}}$ is the natural b.c..

Transverse Material (Cross-Sectional) Symmetry

Recall (p. 137) that the stored-energy function has the form

$$(*) \quad \Phi(\underset{\sim}{r}'; \underset{=}{R}, \underset{=}{R}') = \Psi(\underset{=}{R}^T \underset{\sim}{r}', \underset{=}{R}^T \underset{=}{K} \underset{=}{R}),$$

where $\underset{=}{K} \equiv \underset{=}{R}' \underset{=}{R}^T$.

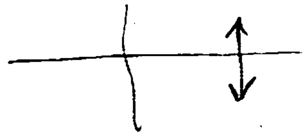
<u>Define</u> $\quad O_c(2) = \{ \underset{=}{Q} \in O(3) : \underset{=}{Q} \underset{\sim}{e}_3 = \underset{\sim}{e}_3 \}$

<u>Note</u>: $O_c(2)$ contains <u>proper</u> <u>rotations</u> $\underset{=}{R}(\theta) \in SO(3)$, where

$$\underset{\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}}{[R(\theta)]} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad 0 \le \theta < 2\pi,$$

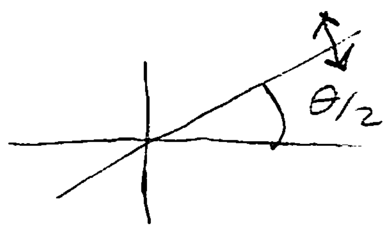
and <u>reflections</u>, e.g., $\underset{=}{E}$ such that $\underset{=}{E}\underset{\sim}{e}_1 = \underset{\sim}{e}_1$, $\underset{=}{E}\underset{\sim}{e}_2 = -\underset{\sim}{e}_2$ and $\underset{=}{E}\underset{\sim}{e}_3 = \underset{\sim}{e}_3$:

$$[E] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$



and more generally $\underset{=}{E}\underset{=}{Q}$:

$$[E][R(\theta)] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

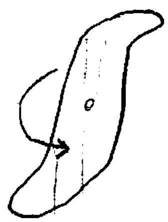


Direct Approach

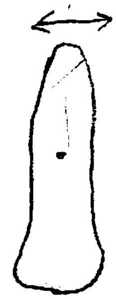
The rod is said to have transverse material (cross-sectional) symmetry $\underline{\underline{Q}} \in O_c(2)$ if

$$(*) \quad \Phi(\underline{r}', \underline{\underline{R}}\,\underline{\underline{Q}}, \underline{\underline{R}}'\underline{\underline{Q}}) = \Phi(\underline{r}', \underline{\underline{R}}, \underline{\underline{R}}').$$

Observe: we allow reflections ($\det \underline{\underline{Q}} = -1$) in our characterization of symmetry. Indeed, we want to distinguish between cross-sections like



$\underline{\underline{R}}(\pi)$ and



$\underline{\underline{E}}\,\underline{\underline{R}}(\pi)$

To see why $(*)$ is appropriate, we again go back to the constrained 3-d point of view:

p. 134

"Experiment ①" $\quad \underline{\underline{F}}^{①} = \underline{\underline{R}}\,\underline{e}_\alpha \otimes \underline{e}_\alpha + (\underline{r}' + X_\alpha \underline{\underline{R}}'\underline{e}_\alpha) \otimes \underline{e}_3$

"Experiment ②" First rotate or reflect rod via $\underline{\underline{Q}} \in O_c(2)$. Then "perform" experiment ①:

Now $\underset{\sim}{f}^{(2)}(\underset{\sim}{X}) = \underset{\sim}{r}(s) + X_\alpha \tilde{\underset{\sim}{d}}_\alpha(s)$ as on p.123,

But now $\tilde{\underset{\sim}{d}}_\alpha = \underset{\approx}{R}\tilde{\underset{\sim}{e}}_\alpha \equiv \underset{\approx}{R}\,\underset{\approx}{Q}\underset{\sim}{e}_\alpha$.

The same calculations as on pp. 133-134 yield

$$\underset{\approx}{F}^{(2)} = \tilde{\underset{\sim}{d}}_\alpha \otimes \underset{\sim}{e}_\alpha + (\underset{\sim}{r}' + X_\alpha \tilde{\underset{\sim}{d}}_\alpha')\otimes \underset{\sim}{e}_3$$
$$= \underset{\approx}{R}\,\underset{\approx}{Q}\,\underset{\sim}{e}_\alpha \otimes \underset{\sim}{e}_\alpha + (\underset{\sim}{r}' + X_\alpha \underset{\approx}{R}'\underset{\approx}{Q}\underset{\sim}{e}_\alpha)\otimes \underset{\sim}{e}_\alpha$$

Summary

$$\overset{(1)}{\underset{\approx}{R}, \underset{\approx}{R}'} \rightarrow \overset{(2)}{\underset{\approx}{R}\,\underset{\approx}{Q}, \underset{\approx}{R}'\underset{\approx}{Q}}$$

Observe that (*) p.150 is equivalent to

$$\int_\Omega W\big(\underset{\approx}{R}\underset{\sim}{e}_\alpha\otimes\underset{\sim}{e}_\alpha + (\underset{\sim}{r}' + X_\alpha \underset{\approx}{R}'\underset{\sim}{e}_\alpha)\otimes\underset{\sim}{e}_3\big)\,dX_1\,dX_2$$
$$= \int_\Omega W\big(\underset{\approx}{R}\,\underset{\approx}{Q}\underset{\sim}{e}_\alpha\otimes\underset{\sim}{e}_\alpha + (\underset{\sim}{r}' + X_\alpha \underset{\approx}{R}'\underset{\approx}{Q}\underset{\sim}{e}_\alpha)\otimes\underset{\sim}{e}_3\big)\,dX_1\,dX_2.$$

where $\hat{\underset{\sim}{X}} = X_\alpha \underset{\sim}{e}_\alpha$ on the left and $\hat{\underset{\sim}{X}} = X_\alpha \underset{\approx}{Q}\underset{\sim}{e}_\alpha$ on the right. $\Rightarrow \underset{\approx}{Q}(\Omega) = \Omega$ is a necessary condition for equality above — otherwise the integral on the right may not even make sense!

Next we want to realize (*) p.149 in the reduced form (*) p.148: Since we have "opened up the door" to $\underset{\approx}{R} \rightarrow \underset{\approx}{R}\,\underset{\approx}{Q}$ with $\det(\underset{\approx}{R}\,\underset{\approx}{Q}) = \det\underset{\approx}{R}\,\det\underset{\approx}{Q} = -1$, we need to be careful in extracting an axial vector as at the bottom of p.141 :