Lecture 19

Now $\underset{\sim}{f}^{(2)}(\underline{X}) = \underset{\sim}{r}(s) + X_\alpha \tilde{\underset{\sim}{d}}_\alpha(s)$ as on p. 123,

<u>But</u> now $\tilde{\underset{\sim}{d}}_\alpha = \underline{\underline{R}}\,\tilde{\underset{\sim}{e}}_\alpha \equiv \underline{\underline{R}}\,\underline{\underline{Q}}\,\underset{\sim}{e}_\alpha$.

The same calculations as on pp. 133–134 yield

$$\underline{\underline{F}}^{(2)} = \tilde{\underset{\sim}{d}}_\alpha \otimes \underset{\sim}{e}_\alpha + (\underset{\sim}{r}' + X_\alpha \tilde{\underset{\sim}{d}}_\alpha') \otimes \underset{\sim}{e}_3$$

$$= \underline{\underline{R}}\,\underline{\underline{Q}}\,\underset{\sim}{e}_\alpha \otimes \underset{\sim}{e}_\alpha + (\underset{\sim}{r}' + X_\alpha \underline{\underline{R}}'\underline{\underline{Q}}\,\underset{\sim}{e}_\alpha) \otimes \underset{\sim}{e}_\alpha$$

<u>Summary</u>

$$\overset{(1)}{\underline{\underline{R}}, \underline{\underline{R}}'} \rightarrow \overset{(2)}{\underline{\underline{R}}\,\underline{\underline{Q}}, \underline{\underline{R}}'\underline{\underline{Q}}}$$

Observe that (*) p. 151 is equivalent to

$$\int_\Omega W(\underline{\underline{R}}\,\underset{\sim}{e}_\alpha \otimes \underset{\sim}{e}_\alpha + (\underset{\sim}{r}' + X_\alpha \underline{\underline{R}}'\underset{\sim}{e}_\alpha) \otimes \underset{\sim}{e}_3)\, dX_1\, dX_2$$

$$= \int_\Omega W(\underline{\underline{R}}\,\underline{\underline{Q}}\,\underset{\sim}{e}_\alpha \otimes \underset{\sim}{e}_\alpha + (\underset{\sim}{r}' + X_\alpha \underline{\underline{R}}'\underline{\underline{Q}}\,\hat{\underset{\sim}{e}}_\alpha) \otimes \underset{\sim}{e}_3)\, dX_1\, dX_2.$$

where $\hat{\underset{\sim}{X}} = X_\alpha \underset{\sim}{e}_\alpha$ on the left and $\hat{\underset{\sim}{X}} = X_\alpha \underline{\underline{Q}}\,\underset{\sim}{e}_\alpha$ on the right. $\Rightarrow \underline{\underline{Q}}(\Omega) = \Omega$ is a necessary condition for equality above — otherwise the integral on the right may not even make sense!

Next we want to realize (*) p. 149 in the reduced form (*) p. 148: Since we have "opened up the door" to $\underline{\underline{R}} \rightarrow \underline{\underline{R}}\,\underline{\underline{Q}}$ with $\det(\underline{\underline{R}}\,\underline{\underline{Q}}) = \underbrace{\det \underline{\underline{R}}}_{1}\,\underbrace{\det \underline{\underline{Q}}}_{-1} = -1$, we need to be careful in extracting an axial vector as at the bottom of p. 141:

<u>Lemma</u> The axial vector of $\mathbf{Q}\mathbf{W}\mathbf{Q}^T$ for $\mathbf{Q} \in O(3)$, $\mathbf{W} \in \text{Skew}(\mathbb{E}^3)$, is given by

$$(\det \mathbf{Q})\, \mathbf{Q} \mathbf{w},$$

where $\text{axial}(\mathbf{W}) = \mathbf{w}$. (Note $\det \mathbf{Q} = \pm 1$)

<u>Pf.</u> From the Lemma on p. 27, we have

$$\mathbf{Q}\mathbf{W}\mathbf{a} = \mathbf{Q}(\mathbf{w} \times \mathbf{a}) = \det \mathbf{Q}\, (\mathbf{Q}\mathbf{w}) \times (\mathbf{Q}\mathbf{a}).$$

Choose $\mathbf{a} = \mathbf{Q}^T \mathbf{b} \Rightarrow$

$$\mathbf{Q}\mathbf{W}\mathbf{Q}^T\mathbf{b} = (\det \mathbf{Q})(\mathbf{Q}\mathbf{w}) \times \mathbf{b} \quad \forall \mathbf{b}. \ \square$$

$\therefore$ (*) p. 151 reads

$$\Phi(\mathbf{r}', \mathbf{R}, \mathbf{R}') = \Psi(\mathbf{R}^T\mathbf{r}', (\det \mathbf{R})\mathbf{R}^T\mathbf{k})$$

Thus (*) p. 152 is equivalent to

$$(*) \quad \Psi(\mathbf{Q}^T\mathbf{R}^T\mathbf{r}', \underbrace{\det(\mathbf{R}\mathbf{Q})}_{\det \mathbf{R}\, \det \mathbf{Q}}\, \mathbf{Q}^T\mathbf{R}^T\mathbf{k}) = \Psi(\mathbf{R}^T\mathbf{r}', \cancel{(\det \mathbf{R})}\mathbf{R}^T\mathbf{k})$$

For real deformations of the rod, $\det \mathbf{R} = 1$.

<u>Claim</u>: If (*) above holds for $\mathbf{Q} \in O_c(2)$, then it also holds for $\mathbf{Q}^T$ (let $\mathbf{R} \to \mathbf{R}\mathbf{Q}^T$).

Also, let $\bar{v} \equiv (v_1, v_2, v_3)$, $\bar{k} \equiv (k_1, k_2, k_3)$

and $\bar{\bar{Q}} \equiv [Q_{ij}] \leftarrow$ rel. $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$.

Then in view of p. 137 & the development above, (*) p. 152 is equivalent to

$$(*)\quad \Upsilon\left(\bar{\bar{Q}}\,\bar{v},\ (\det\bar{\bar{Q}})\,\bar{\bar{Q}}\,\bar{k}\right) = \Upsilon\left(\bar{v}, \bar{k}\right)$$

for $\underline{\underline{Q}} \in \mathcal{O}_c(2)$ a transverse symmetry.

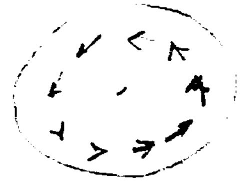
<u>Two interesting cases:</u>

Symmetry group $SO_c(2) = \{ \underline{\underline{R}}(\theta) : 0 \leq \theta < 2\pi \}$

This is typically called <u>hemitropy</u>: (*) above reads

$$\Upsilon\left(\bar{\bar{R}}(\theta)\,\bar{v},\ \bar{\bar{R}}(\theta)\,\bar{k}\right) = \Upsilon\left(\bar{v}, \bar{k}\right) \quad \forall\ 0 \leq \theta < 2\pi$$

(**) rotationally invariant but <u>oriented</u>



$$\Upsilon\left(\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, v_3, \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}, k_3\right)$$

$$= \Upsilon\left(v_1, v_2, v_3, k_1, k_2, k_3\right) \quad \forall\ 0 \leq \theta < 2\pi$$

Thm (Cauchy - rotation invariance with more than one vector argument)

(**) $\Leftrightarrow$

$$\Upsilon(\bar{\nu}, \bar{\kappa}) = \Gamma\left(\nu_1^2 + \nu_2^2, \kappa_1^2 + \kappa_2^2, \nu_1\kappa_1 + \nu_2\kappa_2, \nu_1\kappa_2 - \nu_2\kappa_1, \nu_3, \kappa_3\right)$$

Observe: If $\hat{\nu} = \nu_1 e_1 + \nu_2 e_2$, $\hat{\kappa} = \kappa_1 e_1 + \kappa_2 e_2$, then ~~hemitropic~~

$$(*) \quad \boxed{\Upsilon(\bar{\nu}, \bar{\kappa}) = \Gamma\left(|\hat{\nu}|^2, |\hat{\kappa}|^2, \hat{\nu} \cdot \hat{\kappa}, e_3 \cdot (\hat{\nu} \times \hat{\kappa}), \nu_3, \kappa_3\right)}$$

To quadratic order: assume straight state is "stress" free - 1st order terms vanish

$$\Upsilon(\bar{\nu}, \bar{\kappa}) \doteq \frac{1}{2}\Big\{ A|\hat{\nu}|^2 + B|\hat{\kappa}|^2 + 2C\,\hat{\nu}\cdot\hat{\kappa}$$

$$(**) \quad + 2D\,e_3 \cdot (\hat{\nu} \times \hat{\kappa}) + E(\nu_3 - 1)^2 + F\kappa_3^2$$

$$+ 2G(\nu_3 - 1)\kappa_3 \Big\}$$

$C, D, G \neq 0$ represent coupling coefficients. In particular, G represents a coupling between twist κ_3 and axial strain $\nu_3 - 1$:

<u>Proof</u> (*) p. 155 $\Rightarrow$ hemitropy (**) p. 154 is trivial.

$\overset{?}{\Leftarrow}$

It suffices to show that

$$\Upsilon(\hat{\underset{\sim}{\nu}}^*, \nu_3, \hat{\underset{\sim}{\kappa}}^*, \kappa_3) = \Upsilon(\hat{\underset{\sim}{\nu}}, \nu_3, \hat{\underset{\sim}{\kappa}}, \kappa_3)$$

whenever, $\hat{\underset{\sim}{\nu}}^* \cdot \hat{\underset{\sim}{\nu}}^* = \hat{\underset{\sim}{\nu}} \cdot \hat{\underset{\sim}{\nu}}$, $\hat{\underset{\sim}{\kappa}}^* \cdot \hat{\underset{\sim}{\kappa}}^* = \hat{\underset{\sim}{\kappa}} \cdot \hat{\underset{\sim}{\kappa}}$,

$\hat{\underset{\sim}{\nu}}^* \cdot \hat{\underset{\sim}{\kappa}}^* = \hat{\underset{\sim}{\nu}} \cdot \hat{\underset{\sim}{\kappa}}$, $\underset{\sim}{e}_3 \cdot (\hat{\underset{\sim}{\nu}}^* \times \hat{\underset{\sim}{\kappa}}^*) = \underset{\sim}{e}_3 \cdot (\hat{\underset{\sim}{\nu}} \times \hat{\underset{\sim}{\kappa}})$.

Assume the latter conditions: The first two $\Rightarrow$

$$\hat{\underset{\sim}{\nu}}^* = \underset{\approx}{Q}_1 \hat{\underset{\sim}{\nu}} \qquad [Q_1] = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\hat{Q}_1} & \begin{smallmatrix}0\\0\end{smallmatrix} \\ \hline 0 & 0 & 1 \end{array}\right]$$

$$\hat{\underset{\sim}{\kappa}}^* = \underset{\approx}{Q}_2 \hat{\underset{\sim}{\kappa}} \qquad [Q_2] = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\hat{Q}_2} & \begin{smallmatrix}0\\0\end{smallmatrix} \\ \hline 0 & 0 & 1 \end{array}\right],$$

$\hat{Q}_1$, $\hat{Q}_2$ 2×2 orthogonal matrices. Observe

$$\det \underset{\approx}{Q}_\alpha = \det[\hat{Q}_\alpha] = \pm 1 \qquad \alpha = 1, 2.$$

Then $\hat{\underset{\sim}{\nu}}^* \cdot \hat{\underset{\sim}{\kappa}}^* = \hat{\underset{\sim}{\nu}} \cdot \underset{\approx}{Q}_1^T \underset{\approx}{Q}_2 \hat{\underset{\sim}{\kappa}} = \hat{\underset{\sim}{\nu}} \cdot \hat{\underset{\sim}{\kappa}}$

$$\Rightarrow \underset{\approx}{Q}_1 = \underset{\approx}{Q}_2 \equiv \underset{\approx}{Q} \qquad [Q] = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\hat{Q}} & \begin{smallmatrix}0\\0\end{smallmatrix} \\ \hline 0 & 0 & 1 \end{array}\right],$$

Finally $\underset{\sim}{e}_3 \cdot (\hat{\underset{\sim}{\nu}}^* \times \hat{\underset{\sim}{\kappa}}^*) = \underset{\sim}{e}_3 \cdot (\underset{\approx}{Q} \hat{\underset{\sim}{\nu}} \times \underset{\approx}{Q} \hat{\underset{\sim}{\kappa}})$

$(Qe_3 = e_3)$ $\quad = \underset{\approx}{Q}\underset{\sim}{e}_3 \cdot (\underset{\approx}{Q} \hat{\underset{\sim}{\nu}} \times \underset{\approx}{Q} \hat{\underset{\sim}{\kappa}}) = \det \underset{\approx}{Q}\, \underset{\sim}{e}_3 \cdot (\hat{\underset{\sim}{\nu}} \times \hat{\underset{\sim}{\kappa}})$

$$\therefore \quad e_3 \cdot (\hat{\underset{\sim}{\nu}}^* \times \hat{\underset{\sim}{\kappa}}^*) = \det \underset{=}{Q} \; \underset{=}{e}_3 \cdot (\hat{\underset{\sim}{\nu}} \times \hat{\underset{\sim}{\kappa}})$$

$$\underset{=}{e}_3 \cdot (\hat{\underset{\sim}{\nu}} \times \hat{\underset{\sim}{\kappa}}) \quad \Rightarrow \det \underset{=}{Q} = \det[\hat{Q}] = 1$$

proper $\underset{=}{Q} = \underset{=}{R}(\theta)$

$$\therefore \quad \Upsilon(\hat{\underset{\sim}{\nu}}^*, \nu_3, \hat{\underset{\sim}{\kappa}}^*, \kappa_3) = \Upsilon(\underset{=}{R}(\theta)\hat{\underset{\sim}{\nu}}, \nu_3, \underset{=}{R}(\theta)\hat{\underset{\sim}{\kappa}}, \kappa_3)$$

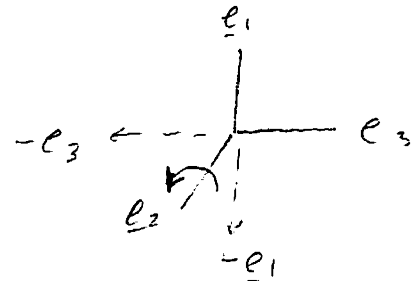
$$= \Upsilon(\bar{\bar{R}}(\theta)\bar{\nu}, \bar{\bar{R}}(\theta)\bar{\kappa})$$

$$\overset{\text{hemitropy}}{=} \Upsilon(\bar{\nu}, \bar{\kappa}). \quad \square$$

If the rod is homogeneous in the straight state (no explicit "s" dependence), then the rod enjoys another symmetry:

As on p. 152, in Experiment ③ we "flip" the rod over by performing a 180° rotation about, say, $\underset{=}{e}_2$: $\underset{=}{G}\underset{=}{e}_1 = -\underset{=}{e}_1$, $\underset{=}{G}\underset{=}{e}_2 = \underset{=}{e}_2$

$\underset{=}{G}\underset{=}{e}_3 = -\underset{=}{e}_3$;



For simplicity, let's imagine doing this at the origin ($s=0$), which need not be the true end of the rod.

The actual deformation we are performing is more complicated than previously in the course: the transformation is <u>global</u>:

$$\underset{\sim}{f}(\underset{=}{G}\,\underset{\sim}{X}) = \underset{\sim}{r}(-s) + X_\alpha \underset{=}{R}(-s)\underset{=}{G}\underset{\sim}{e}_\alpha,$$

i.e., $\underset{\sim}{r}(s) \to \underset{\sim}{r}(-s)$

$$\underset{=}{R}(s) = \underset{=}{R}(-s)\underset{=}{G}$$

$$\underset{=}{G}^T = \underset{=}{G}$$

$$\therefore\ \underset{=}{R}^T_s \underset{\sim}{r}'(s) \to -\underset{=}{G}^T \underset{=}{R}^T_{(s)} \underset{\sim}{r}'(-s) = -\underset{=}{G}\,\underset{=}{R}^T \underset{\sim}{r}'(-s)$$

$$\underset{=}{R}^T \underset{=}{K}\,\underset{=}{R} \to \underset{=}{G}\,\underset{=}{R}^T(-\underset{=}{R}'\underset{=}{G}\,\underset{=}{G}\,\underset{=}{R}^T)\underset{=}{R}\,\underset{=}{G}$$

$$= -\underset{=}{G}\,\underset{=}{R}^T \underset{=}{K}\,\underset{=}{R}\,\underset{=}{G} \quad (\det \underset{=}{G} = 1)$$

$$\underset{=}{R}^T \underset{=}{K} \to -\underset{=}{G}\,\underset{=}{R}^T \underset{=}{K}$$

(evaluate at $s = 0$)

So <u>flip</u> material symmetry reads:

$$\Psi(-\underset{=}{G}\,\underset{=}{R}^T \underset{\sim}{V}, -\underset{=}{G}\,\underset{=}{R}^T \underset{=}{K}) = \Psi(\underset{\sim}{R}^T \underset{\sim}{V}, \underset{=}{R}^T \underset{=}{K})$$

or $\Upsilon(V_1, -V_2, V_3, K_1, -K_2, K_3) = \Upsilon(\bar{V}, \bar{K})$

Combining hemitropy + flip symmetry (p.153):

$$V_1^2 + V_2^2 \to V_1^2 + (-V_2)^2 = V_1^2 + V_2^2$$

$$K_1^2 + K_2^2 \to K_1^2 + K_2^2$$

$$V_1 K_1 + V_2 K_2 \to V_1 K_1 + V_2 K_2$$