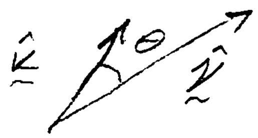
$$\nu_1 k_2 - \nu_2 k_1 \to -\nu_1 k_2 + \nu_2 k_1 = -(\nu_1 k_2 - \nu_2 k_1)$$

Thus, hemitropy + flip symmetry $\Rightarrow$

$$\Gamma(\ldots, -\underset{\sim}{e}_3 \cdot (\hat{\underset{\sim}{\nu}} \times \hat{\underset{\sim}{k}}), \ldots) = \Gamma(\ldots, \underset{\sim}{e}_3 \cdot (\hat{\underset{\sim}{\nu}} \times \hat{\underset{\sim}{k}}), \ldots),$$

which seems to imply that Γ should be a function of $(\underset{\sim}{e}_3 \cdot \hat{\underset{\sim}{\nu}} \times \hat{\underset{\sim}{k}})^2$. However, note that (in the $\underset{\sim}{e}_1 - \underset{\sim}{e}_2$ plane):



$$(\hat{\underset{\sim}{\nu}} \cdot \hat{\underset{\sim}{k}})^2 = |\hat{\nu}|^2 |\hat{k}|^2 \cos^2\theta$$

and

$$(\underset{\sim}{e}_3 \cdot \hat{\underset{\sim}{\nu}} \times \hat{\underset{\sim}{k}})^2 = |\hat{\nu}|^2 |\hat{k}|^2 \sin^2\theta$$

$$\Rightarrow \left(\underset{\sim}{e}_3 \cdot (\hat{\underset{\sim}{\nu}} \times \hat{\underset{\sim}{k}})\right)^2 = |\hat{\underset{\sim}{\nu}}|^2 |\hat{\underset{\sim}{k}}|^2 (1 - \cos^2\theta)$$

$$= |\hat{\underset{\sim}{\nu}}|^2 |\hat{\underset{\sim}{k}}|^2 - (\hat{\underset{\sim}{\nu}} \cdot \hat{\underset{\sim}{k}})^2,$$

which is a combination of other invariants. Accordingly,

$$\boxed{\text{hemi. } (+ \text{ flip} \ \& \ \text{homogeneous}) \Leftrightarrow \tilde{\Gamma}(\underset{\sim}{\nu}, \underset{\sim}{k}) = \Gamma(|\hat{\underset{\sim}{\nu}}|^2, |\hat{\underset{\sim}{k}}|^2, \hat{\underset{\sim}{\nu}} \cdot \hat{\underset{\sim}{k}}, \nu_3, k_3)}$$

Note: to quadratic order, we now have (**) p. 156 with $D = 0$.

<u>Isotropy</u> (+ homogeneity = flip symmetry)

The symmetry group is now.

$$O_c(\underline{z}) = \{ \underline{\underline{R}}(\theta), \underline{\underline{E}}\,\underline{\underline{R}}(\theta) : 0 \le \theta < 2\pi \},$$

i.e., the rod is hemitropic (flip invariant), and

$\det \underline{\underline{E}} = -1$ (see (*) p. 153)

$$\Psi(\bar{\bar{E}}\bar{\nu}, -\bar{\bar{E}}\bar{\kappa}) = \Psi(\bar{\nu}, \bar{\kappa}).$$

Now $\bar{\bar{E}}\bar{\nu} \overset{(p.149)}{=} (\nu_1, -\nu_2, \nu_3)$, $-\bar{\bar{E}}\bar{\kappa} = (-\kappa_1, \kappa_2, -\kappa_3)$.

$$\Rightarrow |\hat{\underset{\sim}{\nu}}|^2 \to |\hat{\underset{\sim}{\nu}}|^2, \; |\hat{\underset{\sim}{\kappa}}|^2 \to |\hat{\underset{\sim}{\kappa}}|^2, \; \hat{\underset{\sim}{\nu}}\cdot\hat{\underset{\sim}{\kappa}} \to -\nu_1\kappa_1 - \nu_2\kappa_2 = -\hat{\underset{\sim}{\nu}}\cdot\hat{\underset{\sim}{\kappa}}$$

and $\kappa_3 \to -\kappa_3$.

$\therefore$ Isotropic (homogeneous = flip symmetric)

$\Longleftrightarrow$

(p. 158)

$$\tilde{\Gamma}(|\hat{\underset{\sim}{\nu}}|^2, |\hat{\underset{\sim}{\kappa}}|^2, -\hat{\underset{\sim}{\nu}}\cdot\hat{\underset{\sim}{\kappa}}, \nu_3, -\kappa_3) = \tilde{\Gamma}(|\hat{\underset{\sim}{\nu}}|^2, |\hat{\underset{\sim}{\kappa}}|^2, \hat{\underset{\sim}{\nu}}\cdot\hat{\underset{\sim}{\kappa}}, \nu_3, \kappa_3)$$

$\Longleftrightarrow$ (claim: $f(x,y) = f(-x,-y) \Leftrightarrow f(x,y) = g(x^2, y^2, xy)$. Pf. Same as p. 157 only easier (try it!).)

(some other function)

$$\Psi(\bar{\nu}, \bar{\kappa}) = \Sigma\left(|\hat{\underset{\sim}{\nu}}|^2, |\hat{\underset{\sim}{\kappa}}|^2, \kappa_3^2, (\hat{\underset{\sim}{\nu}}\cdot\hat{\underset{\sim}{\kappa}})^2, (\hat{\underset{\sim}{\nu}}\cdot\hat{\underset{\sim}{\kappa}})\kappa_3, \nu_3\right).$$

Observe that no coupling of the form $\kappa_3(\nu_3 - 1)$ is now allowed — only via $\kappa_3^2(\nu_3-1)$ (say). In other words "handedness" is now absent. This is more transparent if we compare the quadratic approximation of (*) above, viz.,

$$(*) \quad \Psi(\bar{\nu}, \bar{\kappa}) \doteq \tfrac{1}{2}\{A|\hat{\underset{\sim}{\nu}}|^2 + B|\hat{\underset{\sim}{\kappa}}|^2 + E(\nu_3 - 1)^2 + F\kappa_3^2\}$$

with (**) p. 156 ($D = 0$).

Constraints

Like incompressibility in the 3-d theory, there are some common constraints that are often (typically) employed in rod problems:

1) Unshearability:

Impose the constraint $\underset{\sim}{r}' = \nu_3 \underset{\sim}{d}_3 \quad (\nu_1 = \nu_2 \equiv 0)$

In this case,

$$\underset{\sim}{n} = n_\alpha \underset{\sim}{d}_\alpha + n_3 \underset{\sim}{d}_3 ,$$

where n_1, n_2 primary unknowns (Lagrange multipliers), and the free energy function reduces to

$$W = \Upsilon(\nu_3, \kappa_1, \kappa_2, \kappa_3) \quad , \quad \text{where}$$

$$n_3 = \frac{\partial \Upsilon}{\partial \nu_3} \quad , \quad m_i = \frac{\partial \Upsilon}{\partial \kappa_i} \quad , \quad i = 1, 2, 3$$

Note: n_1 & n_2 are not "elastically" determined by the constitutive law.

2) Inextensibility:

Impose $\quad \underset{\sim}{r}' = \nu_\alpha \underset{\sim}{d}_\alpha + \underset{\sim}{d}_3 \quad (\nu_3 \equiv 1)$

In this case, $\quad W = \Upsilon(\nu_1, \nu_2, \kappa_1, \kappa_2, \kappa_3)$ and

$$\underset{\sim}{n} = \frac{\partial \Psi}{\partial \nu_\alpha} \underset{\sim}{d}_\alpha + n_3 \underset{\sim}{d}_3 \quad ,$$

where n_3 is a primary unknown (Lagrange multiplier).

3) Inextensible, unshearable

This is by far the most popular case (typically with linear elasticities - Love, Landau & Lifschitz, etc):

Impose $\quad \underset{\sim}{r}' = \underset{\sim}{d}_3 \qquad (\nu_1 = \nu_2 \equiv 0, \ \nu_3 \equiv 1)$.

Then $W = \Psi(\kappa)$, and $\underset{\sim}{n}$ itself is a vector-valued Lagrange multiplier.

Exercise (22) The internal potential energy functional for an unshearable, inextensible rod is given by (cf. p. 143)

$$V_i[\underset{\sim}{r}, \underset{\approx}{R}] = \int_0^L \hat{\Psi}(\underset{\approx}{R}^T \underset{\sim}{k})\, ds \,,$$

where $\Psi(\kappa) \equiv \hat{\Psi}(\underset{\approx}{R}^T \underset{\sim}{k})$ is the free energy. Use the Lagrange-multiplier method to enforce the constraint $\underset{\sim}{r}' - \underset{\sim}{d}_3 = \underset{\sim}{r}' - \underset{\approx}{R}\underset{\sim}{e}_3 = \underset{\sim}{0}$, viz., find the stationary conditions for the functional

$$\tilde{V}_i[\underline{r}, \underline{\underline{R}}, \underset{\sim}{n}] = \int_0^L \left[\Psi(\underline{\underline{R}}^T \underline{k}) + \underset{\sim}{\lambda} \cdot (\underline{r}' - \underline{\underline{R}}\,\underline{e}_3) \right] ds$$

Lagrange mult.

As on pp. 147, integrate by parts and compare with pp. 126 to deduce that $\underset{\sim}{\lambda} \equiv \underset{\sim}{n}$.

Now let's compare the quadratic approximations for homogeneous hemitropy (+"flip") (**) p. 156 (w/ $D=0$):

$$\mathcal{V}(\bar{v}, \bar{k}) = \frac{1}{2} \Big\{ A\, \hat{\underset{\sim}{v}} \cdot \hat{\underset{\sim}{v}} + B\, \hat{\underset{\sim}{k}} \cdot \hat{\underset{\sim}{k}} + 2C\, \hat{\underset{\sim}{v}} \cdot \hat{\underset{\sim}{k}} + E(v_3 - 1) + F k_3^2 + 2G(v_3 - 1)k_3 \Big\}$$

w/ homogeneous isotropy (+"flip") (*) p. 161:

$$\mathcal{V}(\bar{v}, \bar{k}) = \frac{1}{2} \Big\{ A\, \hat{\underset{\sim}{v}} \cdot \hat{\underset{\sim}{v}} + B\, \hat{\underset{\sim}{k}} \cdot \hat{\underset{\sim}{k}} + E(v_3 - 1)^2 + F k_3^2 \Big\}$$

<u>Observe</u> in the classical <u>Kirchhoff</u> <u>case</u> quadratic + inextensible unshearable, the two energies are the same!

Strong Ellipticity

Next we investigate the ramifications of strong ellipticity (SE) p. 114 within the context of our rod ~~theory~~ (as a constrained 3-d body):

Claim: If we view hyperelastic rod theory as coming from a constrained 3-d hyperelastic ~~theory~~ (cf. pp. 123, 135, 142-143), then (SE) $\Rightarrow$

$$D^2 \Upsilon(\bar{\nu}, \bar{\kappa}) = \left(\begin{array}{c|c} \dfrac{\partial^2 \Upsilon}{\partial \bar{\nu}^2} & \dfrac{\partial^2 \Upsilon}{\partial \bar{\nu} \partial \bar{\kappa}} \\ \hline \dfrac{\partial^2 \Upsilon}{\partial \bar{\kappa} \partial \bar{\nu}} & \dfrac{\partial^2 \Upsilon}{\partial \bar{\kappa}^2} \end{array} \right)$$

is positive definite.

Pf. To see this, return to the calculations on p. 141-142: E.g.,

$$\frac{\partial \Upsilon}{\partial \nu_i} = \int_\Omega \frac{\partial W}{\partial \underline{\underline{F}}} \Big(\underbrace{\underline{\underline{R}}\left[\underline{e}_\alpha \otimes \underline{e}_\alpha + (\nu_i \underline{e}_i + X_\alpha \kappa_i \underline{e}_i \times \underline{e}_\alpha) \otimes \underline{e}_3\right]}_{} \Big) \cdot (\underbrace{\underline{\underline{R}}\underline{e}_i}_{\underline{d}_i} \otimes \underline{e}_3)\, dA$$

$$\therefore \frac{\partial^2 \Upsilon}{\partial \nu_i \partial \nu_j} = \int_\Omega \underline{d}_i \otimes \underline{e}_3 \cdot \frac{\partial^2 W}{\partial \underline{\underline{F}}^2}(\underline{\underline{F}}_c)\left[\underline{d}_j \otimes \underline{e}_3\right] dA$$