$$\therefore \frac{\partial^2 \Upsilon}{\partial \nu_i \partial \nu_j}(\bar{\nu}, \bar{\kappa}) = \int_\Omega \hat{d}_i \otimes e_3 \cdot \hat{\mathbb{C}}(F_c)[d_j \otimes e_3]\, dA$$

Similarly, (p. 142)

$$\frac{\partial^2 \Upsilon(\bar{\nu}, \bar{\kappa})}{\partial \kappa_j \partial \nu_i} = \int_\Omega \hat{d}_i \otimes e_3 \cdot \mathbb{C}(F_c)[(\hat{d}_j \times I_\alpha \hat{d}_\alpha) \otimes e_3]\, dA,$$

and

$$\frac{\partial^2 \Upsilon}{\partial \kappa_i \partial \kappa_j}(\bar{\nu}, \bar{\kappa}) = \int_\Omega [(\hat{d}_i \times I_\alpha d_\alpha) \otimes e_3] \cdot \mathbb{C}(F_c)[(\hat{d}_j \times I_\beta d_\beta) \otimes e_3] dA$$

Define $\gamma \equiv x_i d_i + y_i \hat{d}_i \times I_\alpha \hat{d}_\alpha$ for scalars x_i, y_i, $i = 1, 2, 3$. Then

$$\underbrace{\int_\Omega (\gamma \otimes e_3) \cdot \mathbb{C}(F_c)[\gamma \otimes e_3]\, dA}_{} \overset{(SE)}{>} 0 \quad \forall \ \gamma \neq 0$$

$$\int_\Omega ((x_i \hat{d}_i + y_i \hat{d}_i \times I_\alpha \hat{d}_\alpha) \otimes e_3) \cdot \hat{\mathbb{C}}(F_c)[(x_j \hat{d}_j + y_j \hat{d}_j \times I_\beta d_\beta) \otimes e_3] dA$$

$$= x_i \frac{\partial^2 \Upsilon}{\partial \nu_i \partial \nu_j} x_j + x_i \frac{\partial^2 \Upsilon}{\partial \nu_i \partial \kappa_j} y_j + y_i \frac{\partial^2 \Upsilon}{\partial \kappa_i \partial \nu_j} x_j$$
$$+ y_i \frac{\partial^2 \Upsilon}{\partial \kappa_i \partial \kappa_j} y_j$$

In matrix form, we have

Standard Euclidean inner product on $\mathbb{R}^6$

$$\left\langle \begin{pmatrix} \bar{X} \\ \hline \bar{Y} \end{pmatrix}, \begin{pmatrix} \frac{\partial^2 \Psi}{\partial \nu^2}(\cdot) & \frac{\partial^2 \Psi}{\partial \nu \partial \kappa}(\cdot) \\ \frac{\partial^2 \Psi}{\partial \kappa \partial \nu}(\cdot) & \frac{\partial^2 \Psi}{\partial \kappa^2}(\cdot) \end{pmatrix} \begin{pmatrix} \bar{X} \\ \hline \bar{Y} \end{pmatrix} \right\rangle > 0$$

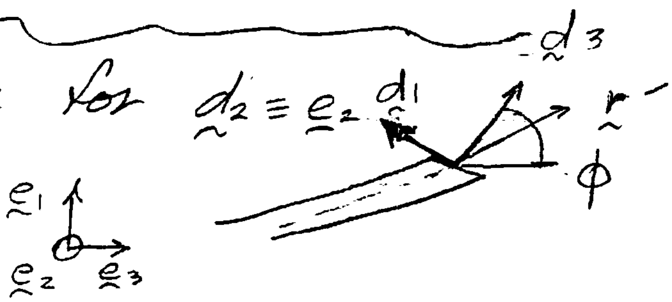
pos-defn

$$\forall \begin{pmatrix} \bar{X} \\ \hline \bar{Y} \end{pmatrix} \in \mathbb{R}^6, \begin{pmatrix} \bar{X} \\ \hline \bar{Y} \end{pmatrix} \neq \begin{pmatrix} 0 \\ \hline 0 \end{pmatrix}, \dots$$

i.e., the Hessian $D^2 \Psi(\nu, \kappa)$ is positive-definite. $\square$

Planar Deformations

Look for $\underset{\sim}{d}_2 \equiv \underset{\sim}{e}_2$



$\underset{\sim}{r}, \hat{\underset{\sim}{b}} \in \text{span}\{\underset{\sim}{e}_1, \underset{\sim}{e}_3\} \Rightarrow \underset{\sim}{r}' \in \text{span}\{\underset{\sim}{e}_1, \underset{\sim}{e}_3\}$

$\hat{\underset{\sim}{g}} \in \text{span}\{\underset{\sim}{e}_2\}$

$\underset{\sim}{R}\,\underset{\sim}{e}_2 = \underset{\sim}{e}_2$

$$\Rightarrow \begin{cases} \underset{\sim}{d}_3(s) = \cos\phi(s)\,\underset{\sim}{e}_3 + \sin\phi(s)\,\underset{\sim}{e}_1 \\ \underset{\sim}{d}_1(s) = -\sin\phi(s)\,\underset{\sim}{e}_3 + \cos\phi(s)\,\underset{\sim}{e}_1 \end{cases}$$

$$\underset{\sim}{d}_3' = \phi'\overbrace{(-\sin\phi\,\underset{\sim}{e}_3 + \cos\phi\,\underset{\sim}{e}_1)}^{\underset{\sim}{d}_1} = \phi'\underset{\sim}{e}_2 \times \underset{\sim}{d}_3$$

$$\underset{\sim}{d}_1' = -\phi'\overbrace{(\cos\phi\,\underset{\sim}{e}_3 + \sin\phi\,\underset{\sim}{e}_1)}^{\underset{\sim}{d}_3} = \phi'\underset{\sim}{e}_2 \times \underset{\sim}{d}_1$$

$$\Rightarrow \underset{\sim}{\kappa} = \phi'\underset{\sim}{e}_2 = \phi'\underset{\sim}{d}_2 \Rightarrow \kappa_2 = \phi', \quad \kappa_1 = \kappa_3 = 0$$

Write $\underset{\sim}{r}' = \xi\,\underset{\sim}{d}_3 + \eta\,\underset{\sim}{d}_1$

$$\Rightarrow \nu_1 = \eta, \quad \nu_2 = 0, \quad \nu_3 = \xi$$

Exercise 23 For an isotropic, homogeneous (flip symmetric) rod, we have the free-energy representation (p. 161)

$$\Upsilon(\bar{\nu}, \bar{\kappa}) = \Sigma\left(\nu_1^2+\nu_2^2, \kappa_1^2+\kappa_2^2, \kappa_3^2, (\nu_1\kappa_1+\nu_2\kappa_2)^2, (\nu_1\kappa_1+\nu_2\kappa_2)\kappa_3, \nu_3\right).$$

For a planar deformation as above $(\kappa_1=\kappa_3=\nu_2\equiv 0)$, show that

$$\underset{\sim}{n} = V\underset{\sim}{d}_1 + N\underset{\sim}{d}_3$$

$$\underset{\sim}{m} = M\underset{\sim}{e}_2,$$

i.e., show that $n_2 = \left.\dfrac{\partial\Upsilon}{\partial\nu_2}\right|_{\kappa_1=\kappa_3=\nu_2=0} = 0$ and

$$m_\alpha = \left.\frac{\partial\Upsilon}{\partial\kappa_\alpha}\right|_{\kappa_1=\kappa_3=\nu_2=0} = 0, \quad \alpha = 1, 3.$$

Recall for equilibrium we require (pp. 126, 129)

$$(*)\quad \begin{cases} \underset{\sim}{n}' + \hat{\underset{\sim}{b}} = \underset{\sim}{0}, \\ \underset{\sim}{m}' + \underset{\sim}{r}' \times \underset{\sim}{n} + \hat{\underset{\sim}{g}} = \underset{\sim}{0}. \end{cases}$$

$$\begin{aligned} \underset{\sim}{n}' &= V'\underset{\sim}{d}_1 + V\underset{\sim}{d}_1' + N'\underset{\sim}{d}_3 + N\underset{\sim}{d}_3' \\ &= V'\underset{\sim}{d}_1 + V(-\phi'\underset{\sim}{d}_3) + N'\underset{\sim}{d}_3 + N\phi'\underset{\sim}{d}_1 \\ &= (V' + N\phi')\underset{\sim}{d}_1 + (N' - V\phi')\underset{\sim}{d}_3 \end{aligned}$$

$$\underset{\sim}{r}' \times \underset{\sim}{n} = (\eta\underset{\sim}{d}_1 + \xi\underset{\sim}{d}_3) \times (V\underset{\sim}{d}_1 + N\underset{\sim}{d}_3) = (\xi V - \eta N)\underset{\sim}{d}_2$$

Finally we assume $\hat{\underset{\sim}{b}} = \hat{b}_1\underset{\sim}{d}_1 + \hat{b}_3\underset{\sim}{d}_3$, $\hat{\underset{\sim}{g}} = \hat{g}\underset{\sim}{e}_2$.

$$\therefore (*) \Rightarrow \quad (**)\quad \begin{cases} V' + N\phi' + \hat{b}_1 = 0 \\ N' - V\phi' + \hat{b}_3 = 0 \\ M' + \xi V - \eta N + \hat{g} = 0 \end{cases}$$

<u>Constitutive hypotheses</u>:

1. <u>Unconstrained</u>: $\Psi(\xi, \eta, \underbrace{K}_{\phi'})$

$$N = \frac{\partial \Psi}{\partial \xi}(\cdot), \quad V = \frac{\partial \Psi}{\partial \eta}(\cdot), \quad M = \frac{\partial \Psi}{\partial K}(\cdot).$$

2. <u>Unshearable</u>: $\eta \equiv 0$ (constraint)

$\Psi(\xi, K)$, V basic unknown

3. <u>Inextensible</u>: $\xi \equiv 1$ (constraint)

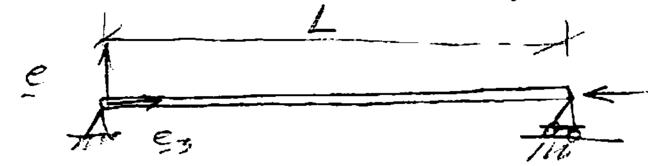
$\Psi(\eta, \kappa)$, N basic unknown

4. <u>Unshearable, inextensible</u>: $\eta \equiv 0$, $\xi \equiv 1$ (constraints)

$\Psi(\kappa)$, V, N basic unknowns

<u>Classical case</u> $\Psi = \frac{1}{2} EI \kappa^2$ (used by Euler 1744)

<u>Ex</u> <u>Column Buckling</u> (Material Model 4)

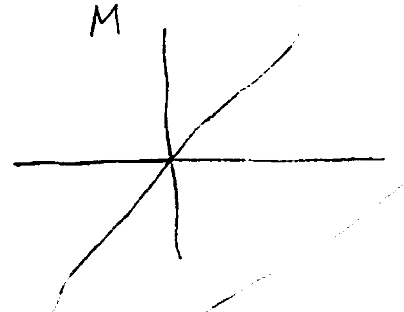


$\eta \equiv 0$, $\xi \equiv 1$

$$M = \frac{d\Psi}{d\kappa}(\kappa) \equiv \hat{M}(\kappa)$$

straight-state "stress-free"

① $\Rightarrow$ $\hat{M}(0) = 0$



moment curvature relation

<u>Properties</u> of $\hat{M}(\cdot)$

② If we return to the representation for isotropic (homogeneous) rods, p168, for this our planar, constrained rod, we have

$$\Psi(\kappa) = \hat{\Sigma}(\kappa^2) \quad \leftarrow \kappa = \kappa_2$$

, and

$$\Upsilon'(\kappa) = 2\hat{\Sigma}'(\kappa^2)\kappa \equiv \hat{M}(\kappa)$$

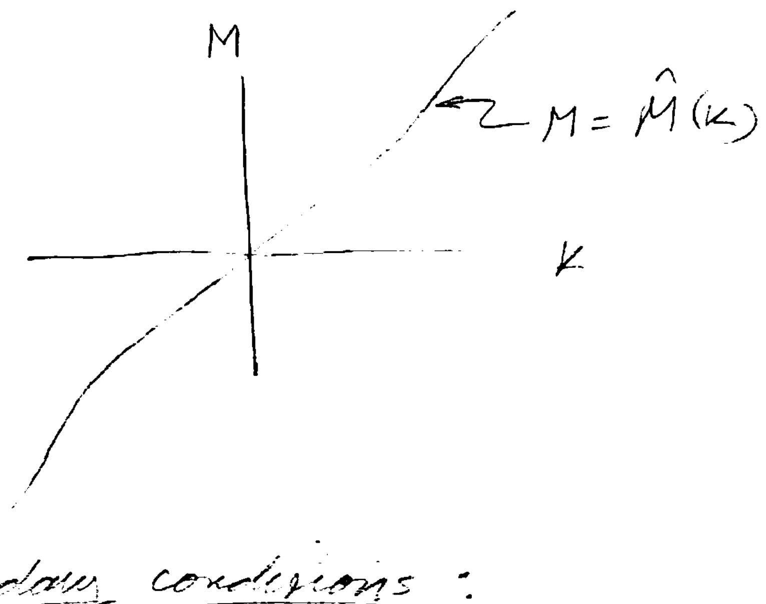
Observe: that $\hat{M}(\cdot)$ is <u>odd</u>.

(3) Returning to "echo" of strong ellipticity in rod theory, pp. 160-162, we see that

$$\frac{d^2\Upsilon}{d\kappa^2}(\kappa) = \hat{M}'(\kappa) > 0 \qquad \underline{\text{monotone increasing}}$$

(4) Finally, it is reasonable to assume

$$\lim_{\kappa \to \infty} \hat{M}(\kappa) = \infty \qquad \underline{\text{growth condition}}$$



<u>Boundary conditions</u>:

$$\underset{\sim}{r}(0) = \underset{\sim}{0}, \quad M(0) = 0$$

$$\underset{\sim}{e}_1 \cdot \underset{\sim}{r}(L) = 0, \quad \underset{\sim}{n}(L) \cdot \underset{\sim}{e}_3 = -P, \quad M(L) = 0$$

Our governing equations (for equilibrium) are (**) p. 169 for zero body force and body couple.

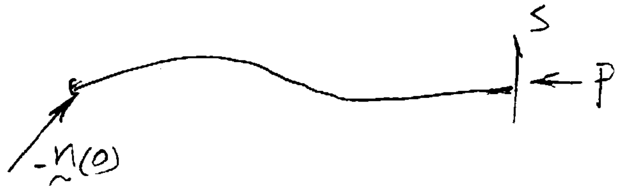
However, we can simplify things by first considering $(*)_1$:

$$\underset{\sim}{n}' = \underset{\sim}{0} \Rightarrow \underset{\sim}{n} = \underset{\sim}{c} \, . \text{ (const)}$$

Write $\underset{\sim}{n}(L) = S\underset{\sim}{e}_1 - P\underset{\sim}{e}_3$ as $\underset{\sim}{n}(L)\cdot\underset{\sim}{e}_3 = -P$

We then return to AME on p. 129 applied to the <u>entire</u> rod:

$$\left(\underset{\sim}{r} \times \underset{\sim}{n} + \underset{\sim}{m}\right)\Big|_0^L = 0 \qquad \text{balance of moments}$$



From the b.c.'s $\underset{\sim}{m}(0) = \underset{\sim}{m}(L) = \underset{\sim}{0}$ and $\underset{\sim}{r}(0) = \underset{\sim}{0}$.

$$\therefore \quad \ell\underset{\sim}{e}_3 \times \left(S\underset{\sim}{e}_1 - P\underset{\sim}{e}_3\right) = \underset{\sim}{0}$$

$$\text{or} \quad S\ell = 0 \qquad \text{for } \ell \neq 0$$

$$\Rightarrow \boxed{S = 0}$$

$$\Rightarrow \quad \underset{\sim}{n}(L) = -P\underset{\sim}{e}_3$$

$$\Rightarrow \quad \underset{\sim}{n} \equiv -P\underset{\sim}{e}_3$$

$$\therefore \quad \underset{\sim}{n} = V\underset{\sim}{d}_1 + N\underset{\sim}{d}_3 = -P\underset{\sim}{e}_3$$

or $V(-\sin\phi\,\underset{\sim}{e}_3 + \cos\phi\,\underset{\sim}{e}_1) + N(\cos\phi\,\underset{\sim}{e}_3 + \sin\phi\,\underset{\sim}{e}_1) = -P\underset{\sim}{e}_3$

$$\Rightarrow \quad N\cos\phi - V\sin\phi = -P$$
$$N\sin\phi + V\cos\phi = 0$$

$$\Rightarrow \quad N = -P\cos\phi \;, \quad V = P\sin\phi$$

((**), p. 167)

Also $\quad M' + V = 0$

$$\therefore \quad M' + P\sin\phi = 0$$

$$\therefore \quad \frac{d}{ds}\left(\hat{M}(\phi')\right) + P\sin\phi = 0$$

$$(*) \qquad \frac{d\hat{M}}{dk}(\phi')\,\phi'' + P\sin\phi = 0$$

$$M(0) = M(L) = 0 \Rightarrow \hat{M}(\phi'(0)) = 0 \Leftrightarrow \phi'(0) = 0$$
$$\hat{M}(\phi'(L)) = 0 \Leftrightarrow \phi'(L) = 0$$

2-point boundary value problem

Note $\quad \underset{\sim}{r}' = \underset{\sim}{d}_3 = \cos\phi\, \underset{\sim}{e}_3 + \sin\phi\, \underset{\sim}{e}_1$

$$\underset{\sim}{r}(0) = \underset{\sim}{0}\,, \quad \underset{\sim}{e}_1 \cdot \underset{\sim}{r}(L) = 0 \qquad \underline{\text{decoupled}} \text{ from } (*)$$

Trivial solution $\phi \equiv 0$ for all P
(straight state)

<u>Linearize about $\phi \equiv 0$:</u> $\quad \phi(s) \to 0 + \alpha h(s)$

$$\frac{d}{d\alpha}\left\{\frac{d\hat{M}}{dk}(\alpha h')[\alpha h''] + P\sin(\alpha h)\right\}\Bigg|_{\alpha=0} = 0$$