Lecture 22

$$\Rightarrow \quad N\cos\phi - V\sin\phi = -P$$
$$N\sin\phi + V\cos\phi = 0$$

$$\Rightarrow \quad N = -P\cos\phi \, , \quad V = P\sin\phi$$

((**), p. 167)

Also $\quad M' + V = 0$

$$\therefore \quad M' + P\sin\phi = 0$$

$$\therefore \quad \frac{d}{ds}\left(\hat{M}(\phi')\right) + P\sin\phi = 0$$

$$(*) \quad \frac{d\hat{M}}{d\kappa}(\phi')\,\phi'' + P\sin\phi = 0$$

$$M(0) = M(L) = 0 \Rightarrow \hat{M}(\phi'(0)) = 0 \Leftrightarrow \phi'(0) = 0$$
$$\hat{M}(\phi'(L)) = 0 \Leftrightarrow \phi'(L) = 0$$

2-point boundary value problem

Note $\underset{\sim}{r}' = \underset{\sim}{d}_3 = \cos\phi\, \underset{\sim}{e}_3 + \sin\phi\, \underset{\sim}{e}_1$

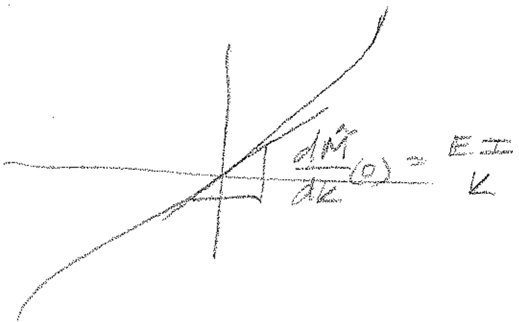
$$\underset{\sim}{r}(0) = \underset{\sim}{0}, \quad \underset{\sim}{e}_1 \cdot \underset{\sim}{r}(L) = 0$$ <u>decoupled from (*)</u>

Trivial solution $\phi = 0$ for all P (straight state)

<u>Linearize about $\phi \equiv 0$:</u> $\phi(s) \to 0 + \alpha h(s)$

$$\frac{d}{d\alpha}\left\{\frac{d\hat{M}}{d\kappa}(\alpha h')[\alpha h''] + P\sin(\alpha h)\right\}\Bigg|_{\alpha=0} = 0$$

$$\Rightarrow \underbrace{\frac{d\hat{M}}{d\kappa}(0)}_{\text{"}EI\text{"}} h'' + Ph = 0$$



$$\Rightarrow (*) \begin{cases} h'' + \lambda h = 0 & , \; 0 < s < L \; , \quad \lambda = \frac{P}{EI} \\ h'(0) = h'(L) = 0 \end{cases}$$

<u>eigenvalue problem</u>:

Nontrivial solutions $\iff \lambda_n = \frac{P_n}{EI} = \frac{n^2\pi^2}{L^2}$

$n = 1, 2, \ldots$

$\underbrace{h_n(s) = \cos\left(\frac{n\pi s}{L}\right)}$ $\qquad \lambda_1 = \frac{\pi^2 EI}{L^2}$ classical Euler buckling load

increment in ϕ away from $\phi \equiv 0$.

For the actual displacement:

$\underset{\sim}{r}' = \underset{\sim}{e}_3$ trivial solution

$\underset{\sim}{u}$ = displacement : $\underset{\sim}{u}(0) = \underset{\sim}{0}$, $\underset{\sim}{e}_1 \cdot \underset{\sim}{u}(L) = 0$

$$\therefore \frac{d}{d\alpha}\left(\underset{\sim}{e}_3 + \alpha \underset{\sim}{u}'\right)\Bigg|_{\alpha=0} = \frac{d}{d\alpha}\left(\cos(\alpha h)\underset{\sim}{e}_3 + \sin(\alpha h)\underset{\sim}{e}_1\right)\Bigg|_{\alpha=0}$$

$$\Rightarrow \underset{\sim}{u}_n' = \underset{\sim}{h}_n = \cos\frac{n\pi s}{L}\underset{\sim}{e}_1 \qquad \underset{\sim}{u}_n(0) = \underset{\sim}{0} \quad \underset{\sim}{e}_1 \cdot \underset{\sim}{u}_n(L) = 0$$

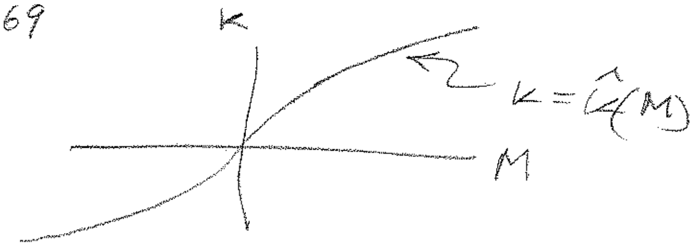
$(\underset{\sim}{u}_n(0) = \underset{\sim}{0})$ $\quad \Rightarrow \underset{\sim}{u}_n = C \sin\frac{n\pi s}{L}\underset{\sim}{e}_1$ $\qquad n = 1$

Exercise (24)

Observe that the eigenvalue problem (*) p. 172 admits the eigenvalue $\lambda_0 = 0$. (a) What is the corresponding eigenfunction? (b) Using the same procedure as at the bottom of p. 172, show that the "buckling mode" for the displacement field is identically zero, i.e., there is no nontrivial solution of the <u>full</u> problem at $\lambda_0 = 0$ ($\Rightarrow P = 0$).

Global Solutions

$M = \hat{M}(\kappa) \underset{\substack{\text{prop. ①-④} \\ \text{pp. 168-169}}}{\Longleftrightarrow} \kappa = \hat{\kappa}(M)$, where



①' $\hat{\kappa}(0) = 0$

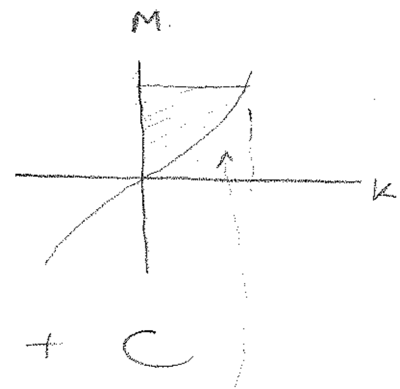
②' $\hat{\kappa}(\cdot)$ is odd

③' $\hat{\kappa}(\ )$ is strictly monotone increasing

④' $\lim_{M \to \infty} \hat{\kappa}(M) \to \infty$.

Consider an equivalent version of the equilibrium equation (p. 171):

$$(*) \quad \begin{cases} \phi' = \hat{\kappa}(M) & (\Leftrightarrow \ \hat{M}(\phi') = M) \\ M' = -P\sin\phi \end{cases}$$



$$\Rightarrow \quad -P\sin\phi\ \phi' = \hat{\kappa}(M)\,M'$$

$$\Rightarrow \quad P\cos\phi = \underbrace{\int_0^M \hat{\kappa}(\sigma)\,d\sigma}_{} + C$$

$\Psi(M) = M\kappa - \Sigma(\kappa)$ compl. energy

or $\quad \Psi(M) - P\cos\phi = C \quad (**)$

<u>Observe:</u> From ①'-③', $\Psi(M)$ is even & convex (strictly so)

"Equilibria" of (*) $\Rightarrow$ $\hat{K}(M) = 0 \Leftrightarrow \underline{M=0}$

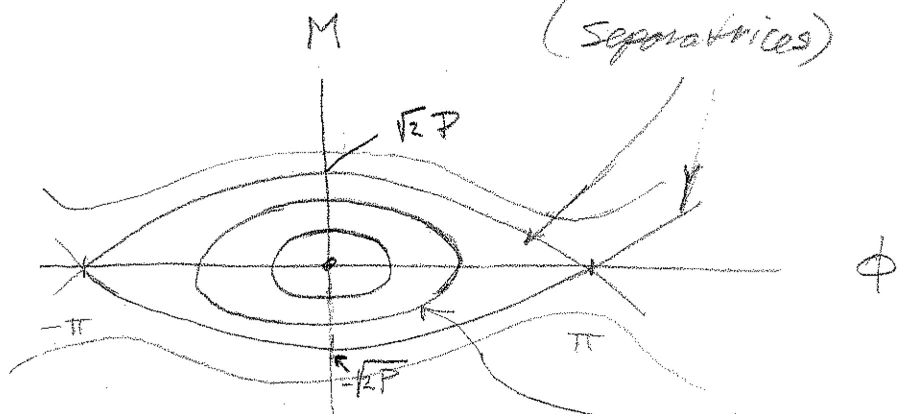
$-P\sin\phi = 0 \Leftrightarrow \phi = 0, n\pi$
$P \neq 0$ $\quad n = 1, 2, \ldots$

<u>phase diagram</u>

— At $M=0, \phi = \pm\pi$, $C = P$

$\therefore \Psi(M) - P\cos\phi = P \Rightarrow$ at $\phi = 0$, $\Psi(M) = 2P$

$M = \pm\sqrt{2P}$



— At $M=0, \phi=0$, $\quad -P = C$

$$\Rightarrow \Psi(M) - P\cos\phi = -P$$

For $\quad -P < C < P$

$\Rightarrow$ closed curves, symmetrical about ϕ & M axes

Of those (<u>integral curves</u> of (*)), only certain ones actually represent actual solutions of the bvp (*) p. 171:

Consider the initial-value problem:

Supplemental Material
not covered in lecture

$$(*)\;\begin{cases} (*)\ \text{p. 176 subject to} \\ \phi(0) = a,\ M(0) = 0 \qquad 0 < a < \pi \end{cases}$$

(skip to p. 182)

Then $\Psi(0)^{\nearrow 0} - P\cos a = C \quad (\Rightarrow -P < C < P)$

$$\Rightarrow \quad \Psi(M) = P(\cos\phi - \cos a)$$

or $\Psi(\hat{M}(\phi')) = P(\cos\phi - \cos a)$

$\phi' \mapsto \Psi(\hat{M}(\phi'))$ convex & even

$$\Rightarrow \quad \underbrace{\Psi(\hat{M}(\phi'))}_{\equiv \phi'\hat{M}(\phi') - \Sigma(\phi')} = \chi((\phi')^2), \text{ where } \chi \text{ is monotone increasing}$$

$$(**) \Rightarrow \quad (\phi')^2 = \chi^{-1}\big(P(\cos\phi - \cos a)\big)$$

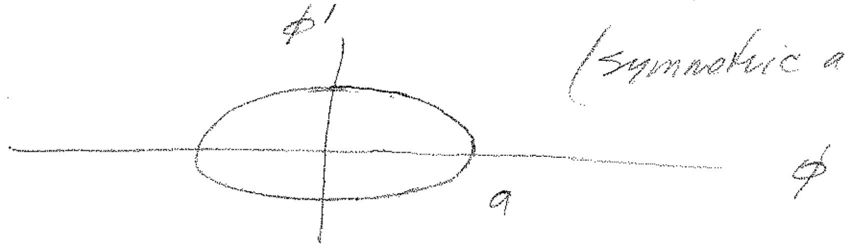
E.g. classical case (Euler) $\hat{M}(\phi') = EI\phi'$

$$\Rightarrow \quad \Psi(M) = \frac{1}{2EI}M^2 = \frac{1}{2EI}(EI)^2(\phi')^2 = \frac{EI}{2}(\phi')^2$$

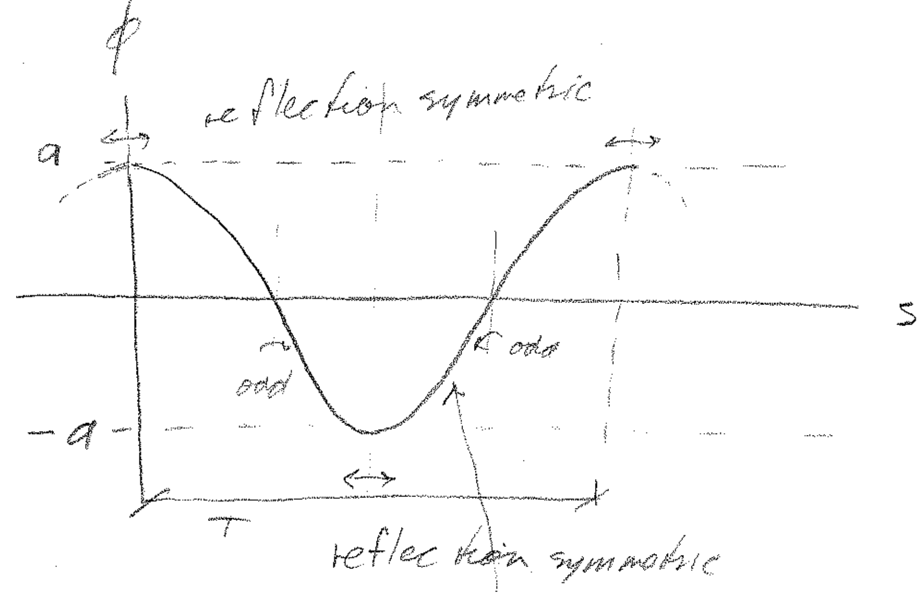
$$(\phi')^2 = \frac{2P}{EI}(\cos\phi - \cos a) = 2\lambda(\cos\phi - \cos a)$$

In any case all solutions are periodic & even at the max/min points:

(symmetric about ϕ', ϕ axes)



We then deduce



"Period" T : $\quad \frac{d\phi}{ds} = \pm\sqrt{\kappa^{-1}(P(\cos\phi - \cos a))}$

$$\therefore\ T = 4\int_0^a \frac{d\phi}{\sqrt{\kappa^{-1}(P(\cos\phi - \cos a))}}$$

In order to satisfy the bvp (*) p.171:

$$n\,\frac{T}{2} = L \qquad n = 1, 2, \dots$$

$$\Rightarrow \quad \frac{L}{2n} = \int_0^a \frac{d\phi}{\sqrt{\kappa^{-1}(P(\cos\phi - \cos a))}}$$

In the Euler case,

$$\frac{L}{2n} = \frac{1}{\sqrt{2\lambda}} \int_0^a \frac{d\phi}{\sqrt{\cos\phi - \cos a}}.$$

If we make the change of variables

$$\sin\frac{\phi}{2} = \sin\frac{a}{2}\sin\sigma ,$$

the dependence on "a" becomes more transparent. Although it works too for the general case, we'll stick to the Euler model, which is much "cleaner".

Note $\phi = 0 \Rightarrow \sigma = 0, \quad \phi = a \Rightarrow \sigma = \pi/2$

$$\sqrt{\cos\phi - \cos a} = \sqrt{\cos\phi - 1 - (\cos a - 1)}$$

$$= \sqrt{2\left(\sin^2\frac{a}{2} - \sin^2\frac{\phi}{2}\right)}$$

$$= \sqrt{2\left(\sin^2\frac{a}{2} - \sin^2\frac{a}{2}\sin^2\sigma\right)}$$

$$= 2\sin\frac{a}{2}\cos\sigma$$

Also $\frac{1}{2}\cos\frac{\phi}{2}\,\phi' = \sin\frac{a}{2}\cos\sigma\,\sigma'$

$$\Rightarrow \phi' = \frac{2}{\cos\frac{\phi}{2}}\sin\frac{a}{2}\cos\sigma\,\sigma'$$

$$= \frac{2}{\sqrt{1-\sin^2\frac{\phi}{2}}}\sin\frac{a}{2}\cos\sigma\,\sigma'$$

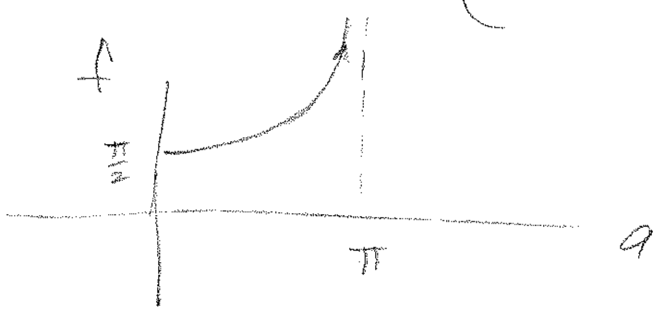
$$= \frac{2\sin\frac{a}{2}\cos\sigma\,\sigma'}{\sqrt{1-\sin^2\frac{a}{2}\sin^2\sigma}}$$

$$(*) \quad \therefore \quad \frac{\sqrt{\lambda}\, L}{2n} = \cancel{\frac{\sqrt{2}}{\sqrt{2}}} \underbrace{\int_0^{\pi/2} \frac{d\sigma}{\sqrt{1-\sin^2\frac{a}{2}\,\sin^2\sigma}}}_{f(a)}$$

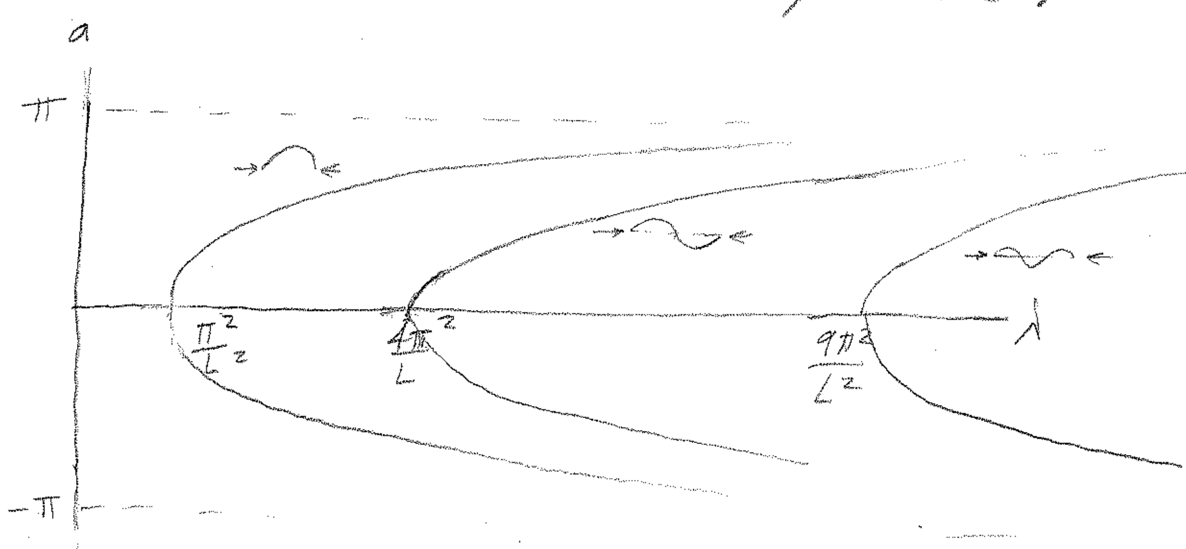
Note: $f(\pi) = \displaystyle\int_0^{\pi/2} \sec\sigma \, d\sigma = \Big(\ln|\sec\sigma + \tan\sigma|\Big)\Big|_0^{\pi/2} \rightarrow +\infty$

Also $f'(a) = \displaystyle\int_0^{\pi/2} \frac{\frac{1}{2}\sin\frac{a}{2}\cos\frac{a}{2}\sin^2\sigma}{\left(1-\sin^2\frac{a}{2}\sin^2\sigma\right)^{3/2}}\, d\sigma > 0$

monotone



$\therefore$ For a given λ, n, (*) above has at most one solution "a". As λ gets larger, more and more "n" become possible:



Ex) Column Buckling (Material Model 2, p. 167)
(Unshearable)

$\eta \equiv 0$ constraint, V Lagrange mult.

$\Psi(\xi, \kappa)$ Free energy

$$N = \frac{\partial \Psi}{\partial \xi}(\xi, \kappa) \equiv \hat{N}(\xi, \kappa) \qquad \left.\begin{aligned} \xi &\equiv d_3 \cdot r' \\ \kappa &\equiv \phi' \end{aligned}\right\} \text{pp. 165-166}$$

$$M = \frac{\partial \Psi}{\partial \kappa}(\xi, \kappa) \equiv \hat{M}(\xi, \kappa)$$

Properties

① straight configuration "stress" free: (assume)

$$\boxed{\hat{N}(1,0) = \hat{M}(1,0) \equiv 0}$$

② Go back to transverse isotropy (p. 161):

$$\Psi(\xi, \kappa) = \Xi(\xi, \kappa^2) \qquad (\xi = \nu_3,\ \kappa = \kappa_2)$$

$$\Rightarrow M = 2\kappa\, D_2 \Xi(\xi, \kappa^2) \equiv \hat{M}(\kappa)$$

$$\therefore \boxed{\kappa \to \hat{M}(\xi, \kappa) \text{ is odd}} \Rightarrow \boxed{\hat{M}(\xi, 0) = 0}$$

Also, $\dfrac{\partial \hat{M}}{\partial \xi}(\xi, \kappa) = 2\kappa \dfrac{\partial}{\partial \xi} D_2 \Xi(\xi, \kappa^2)$

$$\Rightarrow \boxed{\frac{\partial \hat{M}}{\partial \xi}(\xi, 0) \equiv 0 = \frac{\partial^2 \Psi}{\partial \kappa \partial \xi}(\xi, 0) = \frac{\partial \hat{N}}{\partial \kappa}(\xi, 0)}$$

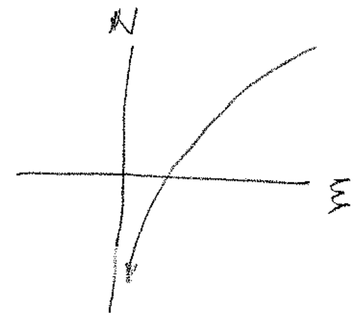
(3) pos-def. (SE) $\Rightarrow$ $\begin{pmatrix} \frac{\partial^2 \psi}{\partial \xi^2} & \frac{\partial^2 \psi}{\partial \xi \partial \kappa} \\ \frac{\partial^2 \psi}{\partial \xi \partial \kappa} & \frac{\partial^2 \psi}{\partial \kappa^2} \end{pmatrix}$ pos. defn.

In part. $\begin{cases} \dfrac{\partial}{\partial \kappa} \hat{M}(\xi,\kappa) = \dfrac{\partial^2 \psi}{\partial \kappa^2}(\xi,\kappa) > 0 \\ \dfrac{\partial}{\partial \xi} \hat{N}(\xi,\kappa) = \dfrac{\partial^2 \psi}{\partial \xi^2}(\xi,\kappa) > 0 \end{cases}$

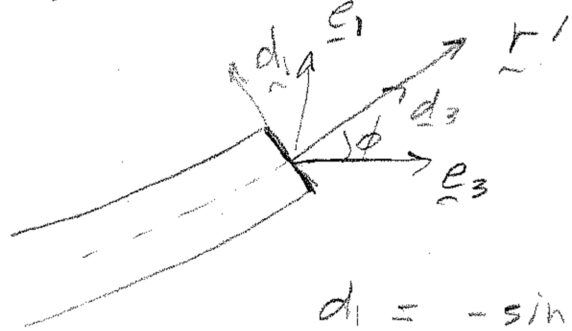
$\Rightarrow$ $\left.\begin{array}{l} \kappa \longmapsto \hat{M}(\xi,\kappa) \\ \xi \mapsto \hat{N}(\xi,\kappa) \end{array}\right\}$ monotone increasing (strictly)

(4) assume growth:

$\left.\begin{array}{l} \lim\limits_{\kappa \to \infty} \hat{M}(\xi,\kappa) \to \infty \\ \lim\limits_{\substack{\xi \to \infty \\ \xi \searrow 0}} \hat{N}(\xi,\kappa) \to \pm\infty \end{array}\right\}$ don't need for local analysis



Now



$r' = \xi d_3$

$d_1 = -\sin\phi\, e_3 + \cos\phi\, e_1$

$d_3 = \cos\phi\, e_3 + \sin\phi\, e_1$

Same considerations as on pp. 172, 173 $\Rightarrow$

(*) $N = \boxed{\hat{N}(\xi,\kappa) = -P\cos\phi}$, $V = P\sin\phi$

Also, (**) p. 169 $\Rightarrow$

$$M' + \xi V = 0$$

$$(*) \quad \therefore \boxed{\frac{d}{ds}\left[\hat{M}(\xi, \phi')\right] + \xi P \sin\phi = 0}$$

Subject to $\quad \underset{\sim}{r}(0) = \underset{\sim}{0} \qquad \underset{\sim}{e}_1 \cdot \underset{\sim}{r}(L) = 0$

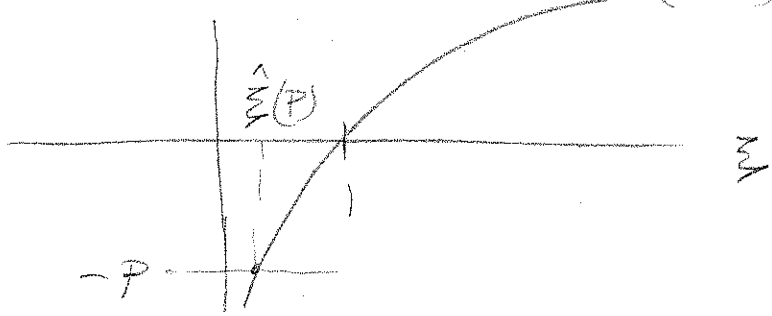
$$\left.\begin{aligned} \hat{M}(\xi(0), \phi'(0)) &= 0 \\ \hat{M}(\xi(L), \phi'(L)) &= 0 \end{aligned}\right\} \quad ② \,\&\, ③ \quad \Leftrightarrow \phi'(0) = \phi'(L) = 0$$

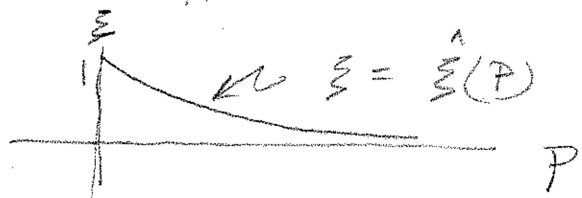
Trivial solution $\quad \boxed{\phi = 0} \quad (\Rightarrow \phi' = 0)$

$\therefore \quad (**) \quad \hat{N}(\xi, 0) = -P \quad \Leftrightarrow \quad \xi = \hat{\xi}(P)$

(*) p. 171

$(P > 0) \qquad 0 < \hat{\xi}(P) < 1$





$\underset{\sim}{r}(s) = \hat{\xi}(P)\, s\, \underset{\sim}{e}_3 \quad$ satisfies field equations

$\phi = 0$

Nontrivial solutions:

$$\underset{\sim}{r}(s) = \hat{\xi}(P)\, s\, \underset{\sim}{e}_3 + \alpha\, \underset{\sim}{u}(s)$$