Lecture 23

$$\phi = \alpha\theta, \quad \xi = \hat{\xi}(P) + \alpha\zeta$$

$$\frac{d}{d\alpha}(\underset{\sim}{r}')\Big|_{\alpha=0} = \underset{\sim}{u}' \qquad (\underset{\sim}{r}' = \xi\,\underline{d}_3)$$

$$= \frac{d}{d\alpha}(\hat{\xi}(P) + \alpha\zeta)\left[\cos(\alpha\theta)\underline{e}_3 + \sin(\alpha\theta)\underline{e}_1\right]\Big|_{\alpha=0}$$

$$= \zeta\,\underline{e}_3 + \hat{\xi}(P)\,\theta\,\underline{e}_1$$

$$\frac{d}{d\alpha}\left[\hat{N}(\hat{\xi}(P)+\alpha\zeta,\ \alpha\theta') = -P\cos(\alpha\theta)\right]\Big|_{\alpha=0}$$

$$\Rightarrow \hat{N}_\xi(\hat{\xi}(P),0)\,\zeta + \cancelto{0}{\hat{N}_\kappa(\hat{\xi}(P),0)}\,\theta' = 0$$

(p. 182)

$$\Rightarrow \boxed{\zeta = 0} \qquad (\therefore\ \underset{\sim}{u}' = \hat{\xi}(P)\,\theta\,\underline{e}_1)$$

no variation

$$\frac{d}{d\alpha}\left[\hat{M}_\kappa(\hat{\xi}(P),\alpha\theta')\,\alpha\theta'' + \hat{\xi}(P)\,P\sin(\alpha\theta)\right]\Big|_{\alpha=0} = 0$$

$$\Rightarrow \hat{M}_\kappa(\hat{\xi}(P),0)\,\theta'' + \hat{\xi}(P)\,P\,\theta = 0$$

$$\theta'(0) = \theta'(1) = 0$$

or
$$\theta'' + \frac{\hat{\xi}(P)}{\hat{M}_\kappa(\hat{\xi}(P),0)}\,P\,\theta = 0$$

Nontrivial $\theta = \cos\frac{n\pi s}{L} \Leftrightarrow$

$$(*) \quad \frac{\hat{\xi}(P)\,P}{\hat{M}_\kappa(\hat{\xi}(P),0)} = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2, \ldots \quad \text{char. eq.}$$

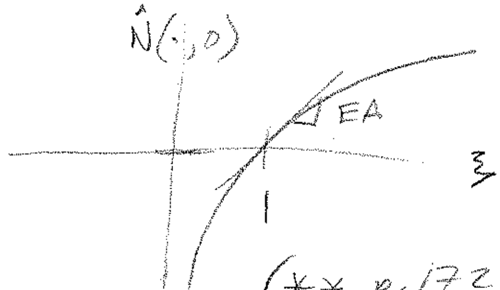
For $n = 1$

$$g(P) \equiv \frac{P\,\hat{\xi}(P)}{\hat{M}_k(\hat{\xi}(P),0)} = \frac{\pi^2}{L^2}$$

Observe: $\hat{\xi}(P)\,,\ \underbrace{\hat{M}_k(\hat{\xi}(P),0)}_{\text{"instaneous bending modulus"}} > 0$

The bifurcation depends crucially upon the behavior of the function $g(P)$:

<u>Ex1</u> Assume $\hat{N}(\xi,0) \equiv EA \ln \xi$



(** p.172)

$$\Rightarrow \quad EA \ln \xi = -P$$

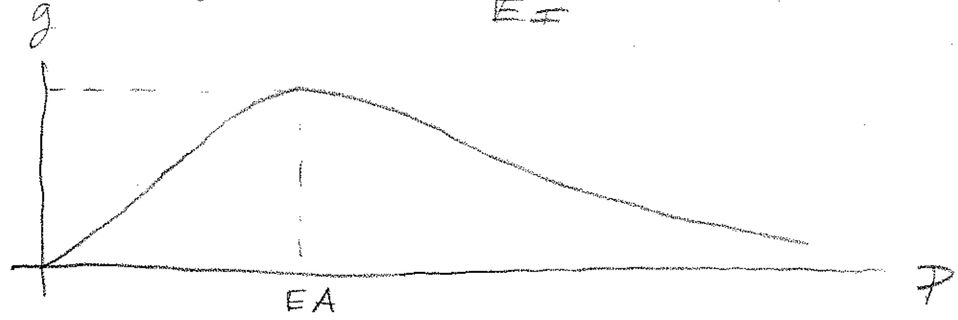
$$\Rightarrow \quad \xi = e^{-P/EA}$$

Choose $\hat{M}_k(\hat{\xi}(P),0) \equiv EI$ (const. "bending" modulus)

Then $g(P) = \dfrac{P e^{-P/EA}}{EI}$

$\dfrac{A}{I}\dfrac{1}{e} \equiv \dfrac{1}{\rho_0^2 e}$

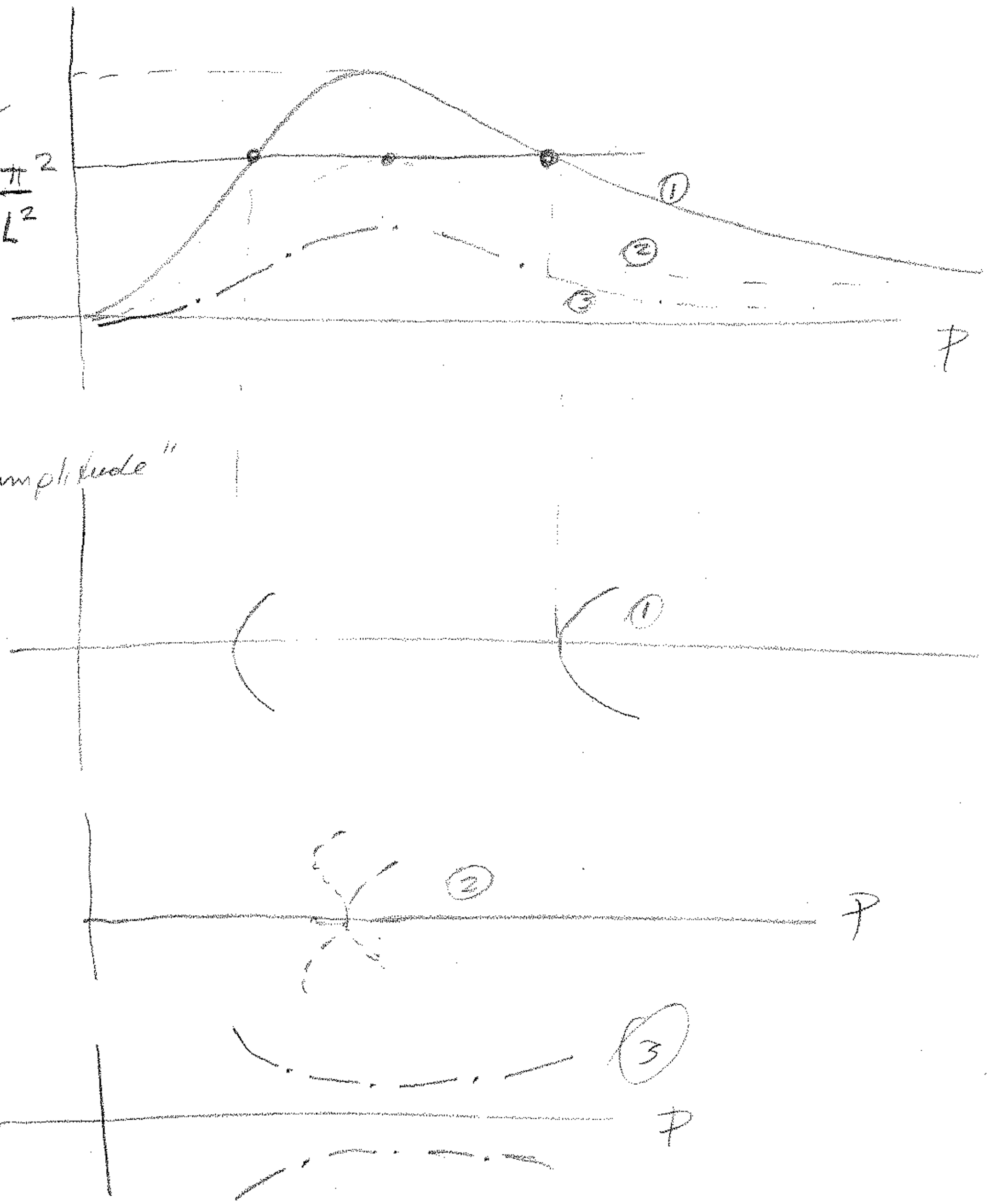
rad gyration



To "solve" the char. eq. (*) p. 173, we compare the graphs

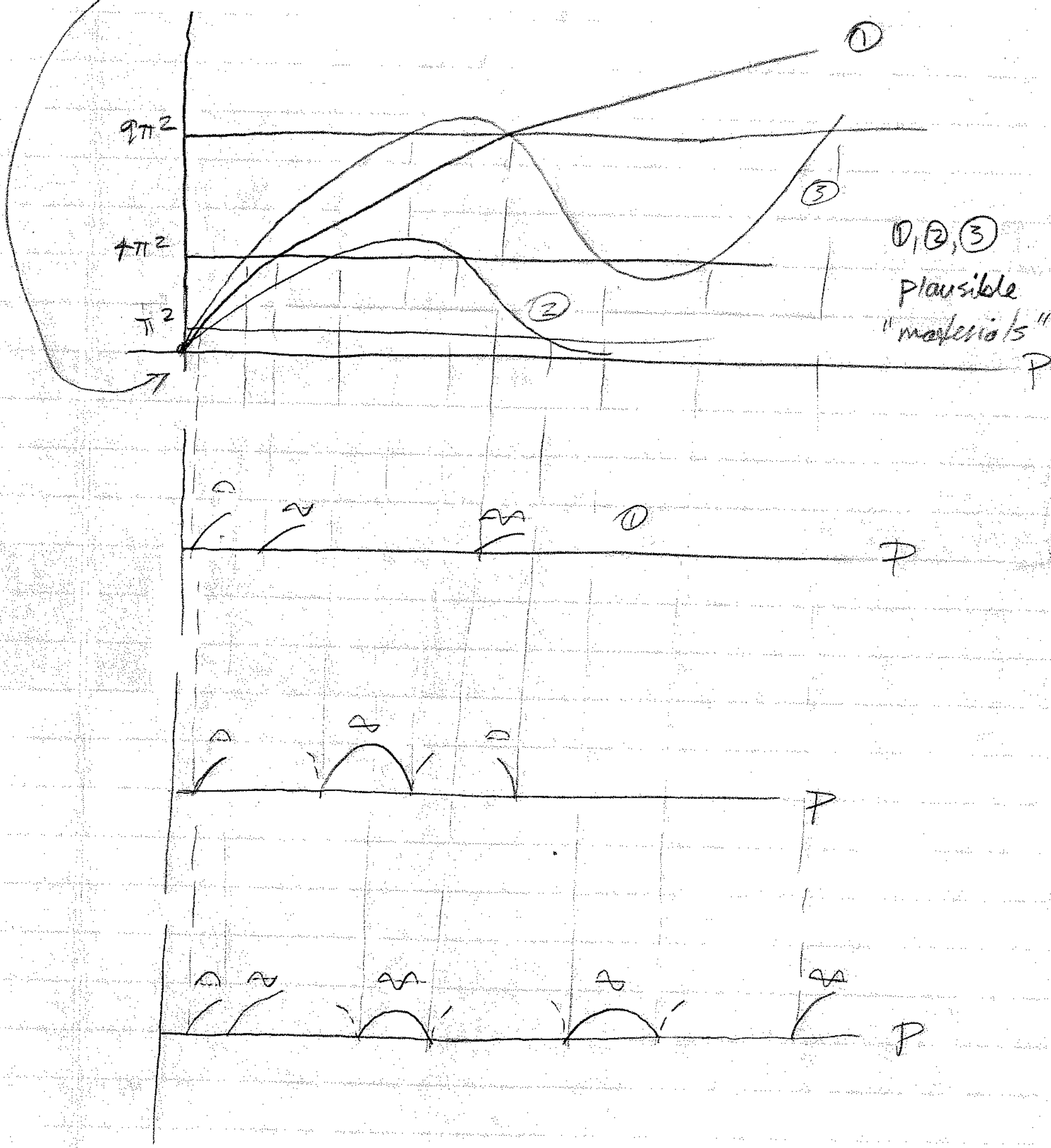
for $\frac{1}{\rho_0^2 e} \le \frac{\pi^2}{L^2}$

or $\frac{\rho_0}{L} \ge \frac{1}{\pi\sqrt{e}} \sim 0.2$



Of course for other materials (cross-sections) other exotic bif. diagrams are possible:

observe $g(0)=0$

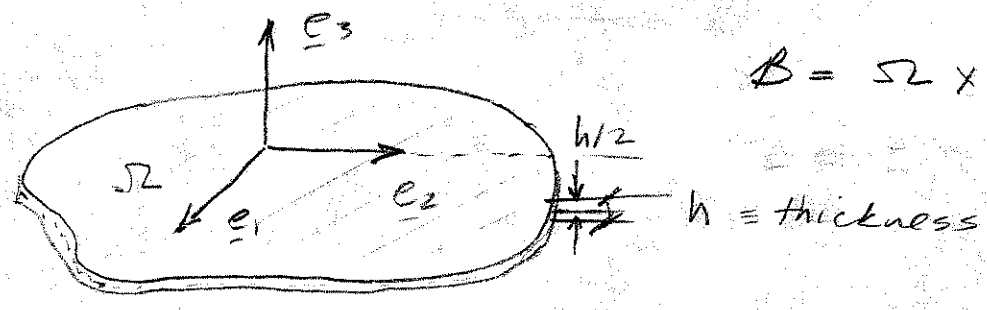


This is what is typically called a *plate* = flat shell

Special Cosserat Theory of Plates (Shells)

Roughly speaking, a plate (shell) is a body for which one dimension – called the thickness – is much smaller than the other two dimensions. As with rods, we will freely bounce back and forth between the direct approach and the constrained (3-d) approach.

For simplicity, we shall assume that the reference configuration is flat with the "middle surface" contained in the plane spanned by $\underset{\sim}{e}_1$ & $\underset{\sim}{e}_2$:



$$\mathcal{B} = \Omega \times \left(-h/2, h/2\right)$$

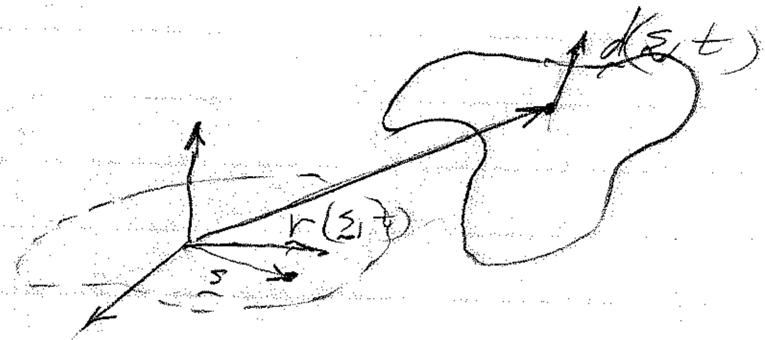
Let $\underset{\sim}{S} = X_\alpha \underset{\sim}{e}_\alpha$ denote the position vector of a material point in Ω. The deformed middle surface (at time t) is specified by

$$\underset{\sim}{S} \mapsto \underset{\sim}{r}(\underset{\sim}{S}, t) \equiv \hat{\underset{\sim}{r}}(X_1, X_2, t),$$

where $\underset{\sim}{r} : \Omega \in \mathbb{E}^2 \to \mathbb{E}^3$. To model the effects of thickness, we attach a director field $\underset{\sim}{d}(\underset{\sim}{s}, t)$ to each material point, which represents the orientation of the "through-thickness fiber" original aligned with $\underset{\sim}{e}_3$ in the refence configuratio

Constrained 3-d motion

$$\underset{\sim}{f}_c(\underline{X}, t) = \underset{\sim}{r}(\underset{\sim}{s}, t) + X_3 \underset{\sim}{d}(\underset{\sim}{s}, t)$$



Then

$$\underset{\sim}{F}_c(\underline{X}, t) = \underline{\nabla} \underset{\sim}{f}(\underline{X}, t)$$

$$= \frac{\partial f_i}{\partial X_\alpha} \underset{\sim}{e}_i \otimes \underset{\sim}{e}_\alpha + \frac{\partial f_i}{\partial X_3} \underset{\sim}{e}_i \otimes \underset{\sim}{e}_3$$

$$= \left(\frac{\partial \tilde{r}_i}{\partial X_\alpha} + X_3 \frac{\partial \tilde{d}_i}{\partial X_\alpha} \right) \underset{\sim}{e}_i \otimes \underset{\sim}{e}_\alpha$$

$$+ \underset{\sim}{d} \otimes \underset{\sim}{e}_3$$

$$= (\tilde{\underset{\sim}{r}}_{,\alpha} + X_3 \tilde{\underset{\sim}{d}}_{,\alpha}) \otimes \underset{\sim}{e}_\alpha + \underset{\sim}{d} \otimes \underset{\sim}{e}_3$$

Aside $\underset{\sim}{s} \mapsto \underset{\sim}{r}$ is differentiable (at $\underset{\sim}{s}$) if

$$\underset{\sim}{r}(\underset{\sim}{s} + \alpha \underset{\sim}{\xi}) = \underset{\sim}{r}(\underset{\sim}{s}) + \underset{\sim}{\nabla} \underset{\sim}{r}(\underset{\sim}{s}) \underset{\sim}{\xi} + o(|\underset{\sim}{\xi}|)$$

as $|\underset{\sim}{\xi}| \to 0$,

where $\underset{\sim}{\nabla} \underset{\sim}{r}(\underset{\sim}{s}) \in L(\mathbb{E}^2, \mathbb{E}^3)$ (for fixed $\underset{\sim}{s}$).

Choose $\underset{\sim}{\xi} = \underset{\sim}{e}_2$. Then

$$\frac{d}{d\alpha} \underset{\sim}{r}(\underset{\sim}{s} + \alpha \underset{\sim}{e}_2)\Big|_{\alpha=0} = \frac{d}{d\alpha} \tilde{\underset{\sim}{r}}(X_1, X_2 + \alpha)\Big|_{\alpha=0}$$

$$\Rightarrow \quad \underset{\sim}{\nabla} \underset{\sim}{r}(\underset{\sim}{s}) \underset{\sim}{e}_2 = \frac{\partial \tilde{\underset{\sim}{r}}}{\partial X_2}(X_1, X_2)$$

$$\underset{\sim}{e}_i \cdot \underset{\sim}{\nabla} \underset{\sim}{r}(\underset{\sim}{s}) \underset{\sim}{e}_2 = \frac{\partial \tilde{r}_i}{\partial X_2}(X_1, X_2)$$

$$\Rightarrow \quad \underset{\sim}{e}_i \cdot \underset{\sim}{\nabla} \underset{\sim}{r}(\underset{\sim}{s}) \underset{\sim}{e}_\alpha = \frac{\partial \tilde{r}_i}{\partial X_\alpha}(X_1, X_2)$$

(*) and $\underset{\sim}{\nabla} \underset{\sim}{r}(s) = \frac{\partial \tilde{r}_i}{\partial X_\alpha}(X_1, X_2)\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_\alpha$

(likewise for $\underset{\sim}{\nabla} \underset{\sim}{d}$)

Returning to (*) p. 188, we have

$$\underset{=}{F}_c = \underset{\sim}{\nabla} \underset{\sim}{r} + X_3 \underset{\sim}{\nabla} \underset{\sim}{d} + \underset{\sim}{d} \otimes \underset{\sim}{e}_3$$

Note: $\underset{\sim}{\nabla} \underset{\sim}{r}(\underset{\sim}{s})$ and $\underset{\sim}{\nabla} \underset{\sim}{d}(\underset{\sim}{s})$ are coordinate-free.