Lecture 24

__Defn__ A _deformation_ of a plate (or flat shell) is a mapping $E^2 \supset \bar{\Omega} \ni \underset{\sim}{s} \mapsto \underset{\sim}{r}(\underset{\sim}{s}), \underset{\sim}{d}(\underset{\sim}{s}) \in \mathbb{E}^3 \times \mathbb{E}^3$. A _motion_ is a one-parameter family (in time) of deformations $\underset{\sim}{r}(\underset{\sim}{s}, t), \underset{\sim}{d}(\underset{\sim}{s}, t)$.

With our direct notation in hand, we can now do the calculation on p. 190 without coordinates: Given $\underset{\sim}{\eta} \in E^3$, write $\hat{\underset{\sim}{\eta}} \equiv \eta_\alpha \underset{\sim}{e}_\alpha$. Then

$$\frac{d}{d\alpha} \underset{\sim}{f}_c(\underset{\sim}{X} + \alpha \underset{\sim}{\eta})\Big|_{\alpha=0} = \frac{d}{d\alpha}\Big[\underset{\sim}{r}(\underset{\sim}{s} + \alpha \hat{\underset{\sim}{\eta}})$$

$$+ \underset{\sim}{e}_3 \cdot (\underset{\sim}{X} + \alpha \underset{\sim}{\eta}) \underset{\sim}{d}(\underset{\sim}{s} + \alpha \hat{\underset{\sim}{\eta}}) \Big]\Big|_{\alpha=0}$$

$$= \left[\underset{\sim}{\nabla} \underset{\sim}{r}(\underset{\sim}{s}) + X_3 \underset{\sim}{\nabla} \underset{\sim}{d}(\underset{\sim}{s}) \right] \hat{\underset{\sim}{\eta}}$$

$$+ \underset{\sim}{e}_3 \cdot \underset{\sim}{\eta}\, \underset{\sim}{d}(\underset{\sim}{s})$$

$$= \left[\underset{\sim}{\nabla} \underset{\sim}{r}(\underset{\sim}{s}) + X_3 \underset{\sim}{\nabla} \underset{\sim}{d}(\underset{\sim}{s}) \right] \hat{\underset{\sim}{\eta}} + \underset{\sim}{d}(\underset{\sim}{s}) \otimes \underset{\sim}{e}_3 \underset{\sim}{\eta}$$

$$\forall \underset{\sim}{\eta}$$

$$\Rightarrow \underset{=}{F}_c(\underset{\sim}{X}) = \underset{\sim}{\nabla} \underset{\sim}{r}(\underset{\sim}{s}) + X_3 \underset{\sim}{\nabla} \underset{\sim}{d}(\underset{\sim}{s}) + \underset{\sim}{d}(\underset{\sim}{s}) \otimes \underset{\sim}{e}_3$$

Remark. Strictly speaking, the notation involved in the expression for $\underset{\approx}{F}_c$ (cf. bottom of p. 191 & bottom of p. 192) is inconsistent because $\nabla \underset{\sim}{r}(\underset{\sim}{s}), \nabla \underset{\sim}{d}(\underset{\sim}{s}) \in L(\mathbb{E}^2, \mathbb{E}^3)$. We can alway get around this via writing, e.g.,

$$\underset{\sim}{r}(\underset{\sim}{s}) \equiv \underset{\sim}{\chi}(\underset{\sim}{s} + 0\underset{\sim}{e}_3) \qquad \underset{\sim}{\chi} : \mathbb{E}^3 \to \mathbb{E}^3.$$

Then $\nabla \underset{\sim}{r}(\underset{\sim}{s}) \cong \nabla \underset{\sim}{\chi}(\underset{\sim}{s}) \in L(\mathbb{E}^3)$, which will always be understood henceforth.

In any case, and in anticipation of things to come, the stored energy function should be of the form

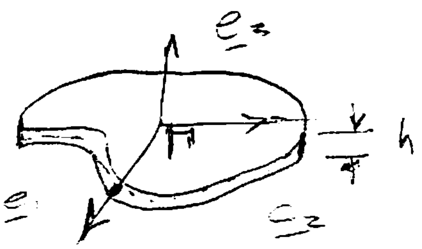
$$\Phi(\nabla \underset{\sim}{r}(\underset{\sim}{s}), \nabla \underset{\sim}{d}(\underset{\sim}{s}), \underset{\sim}{d})$$
$$(*) \qquad \equiv \int_{-h/2}^{h/2} W(\underset{\approx}{F}_c)\, dX_3 .$$

We shall come back to this later.

Instead, we first pursue the through-thickness integration of linear & angular momentum balance (in analogy to what we did with rods):

Linear Momentum Balance

We consider a part $\mathcal{P} \subset \overbrace{\Omega \times (-h/2, h/2)}^{\mathcal{B}}$ as follows: $\mathcal{P} = \Gamma \times (-h/2, h/2)$, $\Gamma \subset \Omega$:



Recall (p. 23)

(LMB) $$\int_{\partial \mathcal{P}} \underset{\approx}{S}\, \underset{\sim}{m}\, ds + \int_{\mathcal{P}} \underset{\sim}{\hat{b}}\, dV = \int_{\mathcal{P}} \rho_0 \frac{d^2 \underset{\sim}{f}}{dt^2}$$

$$\int_{\partial \mathcal{P}} \underset{\approx}{S}\, \underset{\sim}{m}\, ds \overset{(3\text{ parts})}{=} \overbrace{\int_{\Gamma} \underset{\approx}{S}(\underset{\sim}{s} + h/2\, \underset{\sim}{e}_3)\, \underset{\sim}{e}_3\, dA}^{\text{"top"}}$$

$$\overbrace{- \int_{\Gamma} \underset{\approx}{S}(\underset{\sim}{s} - h/2\, \underset{\sim}{e}_3)\, \underset{\sim}{e}_3\, dA}^{\text{"bottom"}}$$

$$+ \oint_{\partial \Gamma} \left[\int_{-h/2}^{h/2} \underset{\approx}{S}(\underset{\sim}{X})\, \underset{\sim}{m}(\underset{\sim}{X})\, dX_3 \right] dl$$

Note: on $\partial\Gamma \times (-h/2, h/2)$, $\underset{\sim}{m}(\underset{\sim}{X}) = \overbrace{\nu_\alpha(\underset{\sim}{s})\, \underset{\sim}{e}_\alpha}^{\underset{\sim}{\nu}(\underset{\sim}{s})} \equiv \underset{\sim}{\nu}(\underset{\sim}{s})$

$\therefore$ The last term above becomes

$$\oint_{\partial \Gamma} \left[\int_{-h/2}^{h/2} \underset{\approx}{S}(\underset{\sim}{s} + X_3 \underset{\sim}{e}_3)\, \underset{\sim}{\nu}(\underset{\sim}{s})\, dX_3 \right] dA.$$

Also, $\int_{\mathcal{B}} \hat{\underset{\sim}{b}}(\underset{\sim}{X})\, dV = \int_{\Gamma} \int_{-h/2}^{h/2} \hat{\underset{\sim}{b}}(\underset{\sim}{s} + X_3 \underset{\sim}{e}_3)\, dX_3\, dA$

$$\int_{\mathcal{B}} \frac{d^2 \underset{\sim}{f}}{dt^2} \rho_0\, dV = \int_{\Gamma} \int_{-h/2}^{h/2} \rho_0(\underset{\sim}{X}) \left[\underset{\sim}{r}_{tt}(\underset{\sim}{s}) + \cancel{X_3 \underset{\sim}{d}(\underset{\sim}{s}, t)}^{\,\to 0} \right] dX_3\, dA$$

$$= \int_{\Gamma} \underset{\sim}{r}_{tt}(\underset{\sim}{s}) \left(\int_{-h/2}^{h/2} \rho_0(\underset{\sim}{s} + X_3 \underset{\sim}{e}_3)\, dX_3 \right) dA$$

body force (per unit Γ)

<u>Define</u>: $\hat{\underset{\sim}{b}}(\underset{\sim}{s}) \equiv \int_{-h/2}^{h/2} \hat{\underset{\sim}{b}}(\underset{\sim}{s} + X_3 \underset{\sim}{e}_3)\, dX_3$

$$+ \underbrace{\underset{\approx}{S}(\underset{\sim}{s} + h/2\, \underset{\sim}{e}_3)\underset{\sim}{e}_3}_{\hat{\underset{\sim}{s}}|_{h/2}} - \underbrace{\underset{\approx}{S}(\underset{\sim}{s} - \tfrac{h}{2}\underset{\sim}{e}_3)\underset{\sim}{e}_3}_{\hat{\underset{\sim}{s}}|_{-h/2}}$$

applied external Piola traction

$$\rho_0(\underset{\sim}{s}) \equiv \int_{-h/2}^{h/2} \rho_0(\underset{\sim}{s} + X_3 \underset{\sim}{e}_3)\, dX_3$$ density (per unit Γ)

<u>Define</u> contact force tensor, per unit undeformed length

$$\underset{\approx}{N}(\underset{\sim}{s}) \equiv \int_{-h/2}^{h/2} \underset{\approx}{S}(\underset{\sim}{s} + X_3 \underset{\sim}{e}_3)\, dX_3 \Big|_{\mathbb{E}^2}$$

Then $\underset{\approx}{N} \in L(\mathbb{E}^2, \mathbb{E}^3)$.

Alternatively, define

$$\underset{\sim}{N}_\alpha(\underset{\sim}{s}) \equiv \int_{-h/2}^{h/2} \underset{\approx}{S}(\underset{\sim}{s} + X_3 \underset{\sim}{e}_3)\underset{\sim}{e}_\alpha\, dX_3,$$ i.e.,

$$\underset{\sim}{N}_\alpha = \underset{\approx}{N}\,\underset{\sim}{e}_\alpha$$

$$\underset{\sim}{e}_i \cdot \underset{\sim}{N}_\alpha = \underset{\sim}{e}_i \cdot \underset{\approx}{N}\,\underset{\sim}{e}_\alpha = N_{i\alpha}$$

$$\Rightarrow \quad \boxed{\underset{\approx}{N} = (\underset{\sim}{e}_i \cdot \underset{\sim}{N}_\alpha)\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_\alpha}$$

$$(*) \qquad \boxed{= \underset{\sim}{N}_\alpha \otimes \underset{\sim}{e}_\alpha}$$

Returning to (LMB) (p. 194), we now have

$$(**) \quad \boxed{\oint_{\partial\Gamma} \underset{\approx}{N}\,\underset{\sim}{\nu}\, dl + \int_\Gamma \hat{\underset{\sim}{b}}\, dA = \int_\Gamma \underset{\sim}{r}_{tt}\, \rho_0\, dA}$$

(LMB)

We wish to use the divergence theorem on the left side in order to get a localized version of (**) above. However, it's not quite in the right form ($\underset{\approx}{N}\underset{\sim}{\nu} \notin \mathbb{E}^2$, in general!).

Write $\quad \underset{\approx}{N} = \underbrace{N_{\alpha\beta}\, \underset{\sim}{e}_\alpha \otimes \underset{\sim}{e}_\beta}_{\tilde{\underset{\approx}{N}}} + N_{3\gamma}\, \underset{\sim}{e}_3 \otimes \underset{\sim}{e}_\gamma$

For the first part, we have the standard

2-d divergence

$$\oint_{\partial\Gamma} \tilde{\underset{\approx}{N}}\,\underset{\sim}{\nu}\, dl = \int_\Gamma \nabla \cdot \tilde{\underset{\approx}{N}}\, dA,$$

where $(\nabla \cdot \underset{=}{N})_\alpha = \frac{\partial N_{\alpha\beta}}{\partial s_\beta}$.

For the second term observe that

$$N_{3\gamma}\, \underset{\sim}{e}_3 \otimes \underset{\sim}{e}_\gamma\, \nu_\alpha\, \underset{\sim}{e}_\alpha = N_{3\gamma}\, \underset{\sim}{e}_3\, \delta_{\gamma\alpha}$$

$$= N_{3\alpha}\, \nu_\alpha\, \underset{\sim}{e}_3$$

Call $\underset{\sim}{h} \equiv N_{3\beta}\, \underset{\sim}{e}_\beta \in \mathbb{E}^2$

Then $\displaystyle\oint_{\partial\Gamma} N_{3\alpha}\, \nu_\alpha\, dl = \oint_{\partial\Gamma} \underset{\sim}{h} \cdot \underset{\sim}{\nu}\, dl$

(vect. div thm in $\mathbb{E}^2$)

$$= \int_\Gamma \underset{\sim}{\nabla} \cdot \underset{\sim}{h}\, dA$$

$$= \int_\Gamma N_{3\beta,\beta}\, dA$$

Putting it all together,

2-d div acting on field in $\mathbb{E}^3$.

$$\oint_{\partial\Gamma} \underset{=}{N}\, \underset{\sim}{\nu}\, dl = \oint_\Gamma \underline{\nabla} \cdot \underset{=}{N}\, dA \,,$$

where $\underline{\nabla} \cdot \underset{=}{N} = N_{i\alpha,\alpha}\, \underset{\sim}{e}_i$. So the local form of (LMB) is:

$$\boxed{\nabla \cdot \underline{\underline{N}} + \hat{\underline{b}} = \rho_0 \underline{r}_{tt}} \quad \text{on } \Omega \text{ (for all } t)$$

In components

$$\frac{\partial N_{i\alpha}}{\partial X_\alpha} + b_i = \rho_0 \frac{d^2 r_i}{dt^2}$$

Finally, observe that (p. 191)

$$\underline{N}_\alpha = \underline{\underline{N}}\, \underline{e}_\alpha = (N_{i\beta}\, \underline{e}_i \otimes \underline{e}_\beta)\, \underline{e}_\alpha$$

$$= N_{i\alpha}\, \underline{e}_i$$

$$\therefore \nabla \cdot \underline{\underline{N}} = \frac{\partial N_{i\alpha}}{\partial X_\alpha} \underline{e}_i = \frac{\partial \underline{N}_\alpha}{\partial X_\alpha}$$

Hence, another (perhaps more common) form of (**) p. 192 reads

$$(*) \quad \underline{N}_{\alpha,\alpha} + \hat{\underline{b}} = \rho_0 \underline{r}_{tt}$$

For the same choice for $\mathcal{B}$ (p. 194), we consider angular momentum balance (pp. 126-127):

Angular Momentum Balance

$$\text{(AMB)} \quad \int_{\partial \mathcal{B}} \underset{\sim}{f} \times \underset{\approx}{S}\, \underset{\sim}{m}\, dS + \int_{\mathcal{B}} \underset{\sim}{f} \times \underset{\sim}{\hat{b}}\, dV$$

$$= \int_{\mathcal{B}} \underset{\sim}{f} \times \underset{\sim}{f}_{tt}\, \rho_0\, dV$$

$$\int_{\partial \mathcal{B}} \underset{\sim}{f} \times \underset{\approx}{S}\, \underset{\sim}{m}\, dS = \int_{\Gamma} \left(\underset{\sim}{r} + \frac{h}{2} \underset{\sim}{d}\right) \times \underset{\approx}{S}\Big|_{X_3 = h/2} \underset{\sim}{e}_3\, dA$$

$$- \int_{\Gamma} \left(\underset{\sim}{r} - \frac{h}{2} \underset{\sim}{d}\right) \times \underset{\approx}{S}\Big|_{X_3 = -h/2} \underset{\sim}{e}_3\, dA$$

$$+ \oint \int_{-h/2}^{h/2} \left(\underset{\sim}{r} + X_3 \underset{\sim}{d}\right) \times \underset{\approx}{S}\, \underset{\sim}{\nu}\, dX_3\, dl .$$

<u>Define</u>: $$\underset{\approx}{M}\, \underset{\sim}{\nu} \equiv \underset{\sim}{d} \times \int_{-h/2}^{h/2} X_3\, \underset{\approx}{S}\, \underset{\sim}{\nu}\, dX_3 \qquad \forall\, \underset{\sim}{\nu} \in \mathbb{E}^2,$$

which is the <u>contact</u> <u>couple</u> tensor per unit undef. length. <u>Note</u> $\underset{\approx}{M} \in L(\mathbb{E}^2, \mathbb{E}^3)$.

Alternatively (just like before, p.195), we may define

$$\underset{\sim}{M}_\alpha \equiv \underset{\sim}{d} \times \int_{-h/2}^{h/2} X_3\, \underset{\approx}{S}\, \underset{\sim}{e}_\alpha\, dX_3 .$$

Then

$$\underset{\approx}{M} = \underset{\sim}{M}_\alpha \otimes \underset{\sim}{e}_\alpha .$$

In any case, the above integral now reads

$$\oint_{\partial\Gamma} (\underset{\sim}{r} \times \underset{=}{N}\underset{\sim}{\nu} + \underset{=}{M}\underset{\sim}{\nu})\, dl$$

Also,
$$\int_{\Theta} \underset{\sim}{f} \times \hat{\underset{\sim}{b}}\, dV = \int_{\Gamma}\int_{-h/2}^{h/2} (\underset{\sim}{r} + X_3 \underset{\sim}{d}) \times \hat{\underset{\sim}{b}}\, dX_3\, dA,$$

and
$$\int_{\Theta} \underset{\sim}{f} \times \underset{\sim}{f}_{tt}\, \rho_0\, dV$$

$$= \int_{\Gamma}\left(\int_{-h/2}^{h/2} (\underset{\sim}{r} + X_3 \underset{\sim}{d}) \times (\underset{\sim}{r}_{tt} + X_3 \underset{\sim}{d}_{tt})\rho_0\, dX_3\right) dA$$

(cross terms fall out)

$$= \int_{\Gamma} \underset{\sim}{r} \times \underset{\sim}{r}_{tt}\, \rho_0(\underset{\sim}{s})\, dA + \int_{\Gamma} \underset{\sim}{d} \times \underset{\sim}{d}_{tt}\left(\int_{-h/2}^{h/2} \rho_0 X_3^2\, dX_3\right) dA$$

$$\equiv \int_{\Gamma} \underset{\sim}{r} \times \underset{\sim}{r}_{tt}\, \rho_0(\underset{\sim}{s})\, dA$$

$$+ \int_{\Gamma} \underset{\sim}{d} \times \underset{\sim}{d}_{tt}\, J_0(\underset{\sim}{s})\, dA \qquad J_0(\underset{\sim}{s}) \equiv \int_{-h/2}^{h/2} X_3^2 \rho_0\, dX_3$$

<u>Define:</u>
$$\hat{\underset{\sim}{g}}(\underset{\sim}{s}) \equiv \underset{\sim}{d} \times \int_{-h/2}^{h/2} X_3\, \hat{\underset{\sim}{b}}\, dX_3$$

$$+ \frac{h}{2}\, \underset{\sim}{d} \times \left(\overbrace{\underset{=}{S}\Big|_{X_3 = h/2} \underset{\sim}{e}_3}^{\hat{\underset{\sim}{\Delta}}|_{h/2}} + \overbrace{\underset{=}{S}\Big|_{X_3 = -h/2} \underset{\sim}{e}_3}^{\hat{\underset{\sim}{\Delta}}|_{-h/2}}\right)$$

← external Piola traction

Then (AMB) now reads:

$$\oint_{\partial\Gamma}\left(\underset{\sim}{r} \times \underset{=}{N}\,\underset{\sim}{\nu} + \underset{=}{M}\,\underset{\sim}{\nu}\right) dl + \int_{\Gamma}\left(\underset{\sim}{r} \times \hat{\underset{\sim}{b}}(\underset{\sim}{s}) + \hat{\underset{\sim}{g}}\right) dA$$

$$= \int_{\Gamma}\left[\underset{\sim}{r} \times \underset{\sim}{r}_{tt}\, \rho_0(\underset{\sim}{s}) + \underset{\sim}{d} \times \underset{\sim}{d}_{tt}\, J_0(\underset{\sim}{s})\right] dA$$

$$\forall\, \Gamma \subset \Omega$$

To get the local form (assuming smoothness), we need to work a bit on the term

$$\underset{\sim}{r} \times \underset{=}{N}\,\underset{\sim}{\nu} = \underset{\sim}{r} \times \underset{=}{N}(\nu_\alpha \underset{\sim}{e}_\alpha)$$

$$= \underset{\sim}{r} \times \underset{\sim}{N}_\alpha \nu_\alpha$$

$$= \underset{\sim}{e}_i\, \varepsilon_{ijk}\, r_j\, N_{k\alpha}\, \nu_\alpha$$

$$\left(\begin{aligned} \underset{\sim}{N}_\alpha &= \underset{\sim}{e}_k \cdot \underset{\sim}{N}_\alpha\, \underset{\sim}{e}_k \\ &= (\underset{\sim}{e}_k \cdot \underset{=}{N}\underset{\sim}{e}_\alpha)\, \underset{\sim}{e}_k \\ &= N_{k\alpha}\, \underset{\sim}{e}_k \end{aligned}\right.$$

$$\varepsilon_{p_1 p_2 p_3} = \begin{cases} 1 & \text{even permutations (e.g., } 1,2,3) \\ -1 & \text{odd permutations (e.g., } 3,2,1) \\ 0 & \text{otherwise} \end{cases}$$

Now $\underset{\sim}{r} \times \underset{=}{N}\,\underset{\sim}{\nu} = \underset{\sim}{e}_i\, \varepsilon_{ijk}\, r_j\, N_{k\alpha}\, \nu_\alpha$

$$\equiv \underset{=}{P}\,\underset{\sim}{\nu},$$

where $\underset{=}{P} = \varepsilon_{ijk}\, r_j\, N_{k\alpha}\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_\alpha$

$$\therefore \int_{\partial\Gamma} (\underset{\sim}{r} \times \underset{=}{N}\,\underset{\sim}{\nu})\, dl = \int_{\partial\Gamma} \underset{=}{P}\,\underset{\sim}{\nu}\, dl$$

$$\overset{(\text{div thm, p. 197})}{=} \int_{\Gamma} \underset{\sim}{\nabla}\cdot \underset{=}{P}\, dA$$

$$(\underset{\sim}{\nabla}\cdot \underset{=}{P})_i = \frac{\partial P_{i\alpha}}{\partial X_\alpha}$$

$$= \varepsilon_{ijk}\left(r_{j,\alpha} N_{k\alpha} + r_j N_{k\alpha,\alpha} \right)$$

$$= \int_{\Gamma} \underset{\sim}{e}_i \frac{\partial P_{i\alpha}}{\partial X_\alpha}\, dA$$

$$\overset{(\text{p. 201})}{=} \int_{\Gamma} \left(\underset{\sim}{r}_{,\alpha} \times \underset{\sim}{N}_\alpha + \underset{\sim}{r} \times \underset{\sim}{\nabla}\cdot \underset{=}{N} \right) dA$$

$\therefore$ (AMB) $\Rightarrow$

$$\int_{\Gamma} \Big(\underset{\sim}{r}_{,\alpha} \times \underset{\sim}{N}_\alpha + \cancel{\underset{\sim}{r} \times (\underset{\sim}{\nabla}\cdot \underset{=}{N}) + \underset{\sim}{r} \times \underset{\sim}{b}} + \overset{\text{div thm}}{\underset{\sim}{\nabla}\cdot \underset{=}{M}} + \hat{\underset{\sim}{g}} \Big) dA$$

(LMB)

$$= \int_{\Gamma} \left(\cancel{\underset{\sim}{r} \times \underset{\sim}{r}_{tt}\, \rho_0} + \underset{\sim}{d} \times \underset{\sim}{d}_{tt}\, J_0 \right) dA$$

$$\forall\, \Gamma \subset \Omega,$$

$$\Rightarrow \quad (*) \quad \boxed{\underset{\sim}{\nabla}\cdot \underset{=}{M} + \underset{\sim}{r}_{,\alpha} \times \underset{\sim}{N}_\alpha + \hat{\underset{\sim}{g}} = J_0\, \underset{\sim}{d} \times \underset{\sim}{d}_{tt}}$$