Lecture 25

$$\therefore \int_{\partial\Gamma} (\underset{\sim}{r} \times \underset{\approx}{N}\underset{\sim}{\nu})\, dl \equiv \int_{\partial\Gamma} \underset{\approx}{P}\underset{\sim}{\nu}\, dl$$

(div thm, p.197)

$$= \int_{\Gamma} \underset{\sim}{\nabla} \cdot \underset{\approx}{P}\, dA$$

$$(\underset{\sim}{\nabla} \cdot \underset{\approx}{P})_i = \frac{\partial P_{i\alpha}}{\partial X_\alpha}$$

$$= \varepsilon_{ijk}\left(r_{j,\alpha} N_{k\alpha} + r_j N_{k\alpha,\alpha}\right)$$

$$= \int_{\Gamma} \underset{\sim}{e}_i \frac{\partial P_{i\alpha}}{\partial X_\alpha}\, dA$$

(p. 201)

$$= \int_{\Gamma} \left(\underset{\sim}{r}_{,\alpha} \times \underset{\sim}{N}_\alpha + \underset{\sim}{r} \times \underset{\sim}{\nabla} \cdot \underset{\approx}{N}\right) dA$$

$\therefore$ (AMB) $\Rightarrow$

div thm

$$\int_{\Gamma} \left(\underset{\sim}{r}_{,\alpha} \times \underset{\sim}{N}_\alpha + \cancel{\underset{\sim}{r} \times (\underset{\sim}{\nabla} \cdot \underset{\approx}{N}) + \underset{\sim}{r} \times \hat{\underset{\sim}{b}}} + \underset{\sim}{\nabla} \cdot \underset{\approx}{M} + \hat{\underset{\sim}{g}}\right) dA$$

(LMB)

$$= \int_{\Gamma} \left(\cancel{\underset{\sim}{r} \times \underset{\sim}{r}_{tt} \rho_0} + \underset{\sim}{d} \times \underset{\sim}{d}_{tt} J_0\right) dA$$

$$\forall\, \Gamma \subset \Omega,$$

$$\Rightarrow \quad (*) \quad \boxed{\underset{\sim}{\nabla} \cdot \underset{\approx}{M} + \underset{\sim}{r}_{,\alpha} \times \underset{\sim}{N}_\alpha + \hat{\underset{\sim}{g}} = J_0\, \underset{\sim}{d} \times \underset{\sim}{d}_{tt},}$$

where (as before) $(\underset{\sim}{\nabla} \cdot \underset{\approx}{M})_i = \dfrac{\partial M_{i\alpha}}{\partial X_\alpha}$.

Alternatively, we may express (*) p. 202 as

$$(*) \quad \boxed{\underset{\sim}{M}_{\alpha,\alpha} + \underset{\sim}{r}_{,\alpha} \times \underset{\sim}{N}_\alpha + \hat{\underset{\sim}{g}} = J_0 \, \underset{\sim}{d} \times \underset{\sim}{d}_{tt}}$$

Remark. Compare (*) above to (*) on p. 129 for rods.

Exercise 25 A membrane is a plate that incapable of sustaining couples - internally & externally - and rotational inertia, i.e., $\underset{\approx}{M} \equiv \underset{\approx}{0}$, $J_0 \equiv 0$, $\hat{\underset{\sim}{g}} \equiv \underset{\sim}{0}$. (a) Show that (*) above then implies that $\underset{\sim}{a}(\underset{\sim}{s}) \cdot \underset{\sim}{N}_\alpha(\underset{\sim}{s}) = 0$, $\alpha = 1, 2$, for any $\underset{\sim}{a}(\underset{\sim}{s})$ s.t. $\underset{\sim}{a}(\underset{\sim}{s}) \cdot \underset{\sim}{r}_{,\alpha}(\underset{\sim}{s}) = 0$, $\alpha = 1, 2$, i.e., $\underset{\sim}{a}(\underset{\sim}{s})$ is $\perp$ to the tangent plane at $\underset{\sim}{r}(\underset{\sim}{s})$.

Hint: Write $\underset{\sim}{a}_1 \equiv \underset{\sim}{r}_{,1}$, $\underset{\sim}{a}_2 \equiv \underset{\sim}{r}_{,2}$, which span the tangent plane to the deformed surface at $\underset{\sim}{s}$. Then write

$$\underset{\sim}{N}_\beta = N_\beta^\alpha \underset{\sim}{a}_\alpha + N_\beta^3 \underset{\sim}{a} \quad , \beta = 1, 2.$$

(b) Deduce also that $N_1^2 = N_2^1$.

Alternative Formulation: Director Momentum Balance

Recall from p. 199,

$$\underset{\sim}{M}_\alpha \equiv \underset{\sim}{d} \times \int_{-h/2}^{h/2} X_3 \underset{\approx}{S} \underset{\sim}{e}_\alpha \, dX_3$$

We now define

$$\underset{\approx}{\mu} \underset{\sim}{\nu} \equiv \int_{-h/2}^{h/2} X_3 \underset{\approx}{S} \underset{\sim}{\nu} \, dX_3 \quad \forall \underset{\sim}{\nu} \in \mathbb{E}^2.$$

Observe $\underset{\approx}{\mu} \in L(\mathbb{E}^2; \mathbb{E}^3)$. We then call

$$\underset{\sim}{\mu}_\alpha \equiv \underset{\approx}{\mu} \underset{\sim}{e}_\alpha \quad \left(\begin{array}{l}\text{similarly to } \underset{\approx}{N} \,\&\, \underset{\sim}{N}_\alpha \\ \text{and } \underset{\approx}{M} \,\&\, \underset{\sim}{M}_\alpha\end{array}\right)$$

Then $(*)$ $\underset{\sim}{M}_\alpha = \underset{\sim}{d} \times \underset{\sim}{\mu}_\alpha$. (p. 199)

Similarly, from p. 200, we have the body couple

$$\hat{\underset{\sim}{g}} \equiv \underset{\sim}{d} \times \left[\int_{-h/2}^{h/2} X_3 \hat{\underset{\sim}{b}} \, dX_3 + \frac{h}{2} \left(\hat{\underset{\sim}{\Delta}} \Big|_{X_3 = h/2} + \hat{\underset{\sim}{\Delta}} \Big|_{X_3 = -h/2} \right) \right]$$

We define the quantity in brackets $[\ ] \equiv \hat{\underset{\sim}{l}}$, i.e.,

$$\hat{\underset{\sim}{g}} = \underset{\sim}{d} \times \hat{\underset{\sim}{l}}.$$

To motivate a balance law involving $\underset{\sim}{\mu}_\alpha$ and $\hat{\underset{\sim}{l}}$, we return again to the part $\mathcal{P} = \Gamma \times (-h/2, h/2)$ from before (p. 194) and start with the <u>local</u> form of (LMB) (p. 28).

$$\nabla \cdot \underline{\underline{S}} + \hat{\underline{b}} = \rho_0 \underline{f}_{tt}$$

We again substitute the ansatz $\underline{f}_c$ (p. 190), multiply the equation above by X_3 and then integrate over $\mathcal{B}$:

(*)
$$\int_\Gamma \int_{-h/2}^{h/2} \left(X_3 \nabla \cdot \underline{\underline{S}} + X_3 \rho_0 \hat{\underline{b}} \right) dX_3 \, dA$$

$$= \int_\Gamma \int_{-h/2}^{h/2} \left(\cancelto{0}{X_3 \underline{r}_{tt}} + X_3^2 \underline{d}_{tt} \right) \rho_0 \, dX_3 \, dA$$

Observe $\nabla \cdot (X_3 \underline{\underline{S}}) = X_3 \nabla \cdot \underline{\underline{S}} + \underline{\underline{S}} \underline{e}_3$

$$\left(\frac{d}{dX_j} (X_3 S_{ij}) = X_3 \frac{\partial S_{ij}}{\partial X_j} + \cancelto{S_{i3}}{\delta_{3j} S_{ij}} \right)$$

$$\therefore \int_\Gamma \int_{-h/2}^{h/2} \left(\nabla \cdot (X_3 \underline{\underline{S}}) - \underline{\underline{S}} \underline{e}_3 \right) dX_3 \, dA$$

(p. 195)

$$+ \int_\Gamma \int_{-h/2}^{h/2} X_3 \hat{\underline{b}} \rho_0 \, dX_3 \, dA = \int_\Gamma \underline{d}_{tt} J_0 \, dA$$

Div thm $\Rightarrow$

$$\int_{\partial \mathcal{B}} X_3 \underline{\underline{S}} \underline{m} \, dA - \int_\Gamma \left(\int_{-h/2}^{h/2} \underline{\underline{S}} \underline{e}_3 \, dX_3 \right) dA$$

$$+ \int_\Gamma \int_{-h/2}^{h/2} X_3 \hat{\underline{b}} \rho_0 \, dX_3 \, dA = \int_\Gamma \underline{d}_{tt} J_0 \, dA$$

$$\int_{\partial \mathcal{B}} X_3 \, \underline{\underline{S}} \, \underline{m} \, dA = \int_{\Gamma} h/2 \, \underline{\underline{S}} \Big|_{X_3 = h/2} \underline{e}_3 \, dA$$

$$+ \int_{\Gamma} (+h/2) \, \underline{\underline{S}} \Big|_{X_3 = -h/2} \underline{e}_3 \, dA$$

$$+ \oint_{\partial \Gamma} \underbrace{\left(\int_{-h/2}^{h/2} X_3 \, \underline{\underline{S}} \, \underline{\nu} \, dX_3 \right)}_{\underline{\underline{M}} \, \underline{\nu}} dl$$

Finally, we define the stress resultant

$$\underset{\sim}{\sigma} \equiv \int_{-h/2}^{h/2} \underline{\underline{S}} \, \underline{e}_3 \, dX_3 .$$

In view of the definition of $\hat{\underset{\sim}{l}}$ on p. 204, then the balance law (*) p. 205 now reads

$$\oint_{\partial \Gamma} \underline{\underline{M}} \, \underline{\nu} \, dl - \int_{\Gamma} \underset{\sim}{\sigma} \, dA$$

$$+ \int_{\Gamma} \hat{\underset{\sim}{l}} \, dA = \int_{\Gamma} \underline{d}_{tt} \, J_0 \, dA$$

As before, the divergence ~~theorem~~ and the usual localization argument then yield or $\underset{\sim}{M}_{\alpha,\alpha}$

$$(*) \qquad \boxed{\nabla \cdot \underset{=}{\mu} - \underset{\sim}{\sigma} + \underset{\sim}{\hat{\ell}} = J_0 \underset{\sim}{d}_{tt}}$$

<u>Question</u>: What is the connection with (AMB)? That is, since

$$\underset{\sim}{M}_\alpha = \underset{\sim}{d} \times \underset{=}{\mu}_\alpha \text{ and } \underset{\sim}{\hat{g}} = \underset{\sim}{d} \times \underset{\sim}{\hat{\ell}},$$

there must be some redundancy here. Let's take the point of view that (*) above holds. Then (AMB) implies (p. 195)

$$\oint_{\partial\Gamma} [\underset{\sim}{r} \times \underset{=}{N} \underset{\sim}{\nu} + (\underset{\sim}{d} \times \underset{=}{\mu} \underset{\sim}{\nu}] \, d\ell$$

$$+ \int_\Gamma [\cancel{\underset{\sim}{r} \times \underset{\sim}{\hat{b}}} + \underset{\sim}{d} \times \underset{\sim}{\hat{\ell}}] \, dA$$

$$(**) \qquad = \int_\Gamma (\cancel{\underset{\sim}{r} \times \underset{\sim}{r}_{tt} \rho_0} + \underset{\sim}{d} \times \underset{\sim}{d}_{tt} J_0) \, dA$$

Recall

$$\oint_{\partial\Gamma} \underset{\sim}{r} \times \underset{=}{N} \underset{\sim}{\nu} \, d\ell \overset{\text{(pp. 201-202)}}{=} \int_\Gamma (\underset{\sim}{r}_{,\alpha} \times \underset{\sim}{N}_\alpha + \underbrace{\cancel{\underset{\sim}{r} \times \nabla \cdot \underset{=}{N}}}_{\text{(LMB)}}) \, dA$$

Moreover, by those same arguments (pp. 201-202), we deduce

$$\int_{\partial\Gamma} \underset{\sim}{d} \times \underset{=}{\mu} \underset{\sim}{\nu} \, d\ell = \int_\Gamma [\underset{\sim}{d}_{,\alpha} \times \underset{\sim}{\mu}_\alpha + \underset{\sim}{d} \times (\nabla \cdot \underset{=}{\mu})] \, dA$$

So the local form of (**) p. 207 (AMB) reads

$$\underset{\sim}{r}_{,\alpha} \times \underset{\sim}{N}_\alpha + \underset{\sim}{d}_{,\alpha} \times \underset{\sim}{\mu}_\alpha + \cancel{\underset{\sim}{d} \times (\nabla \cdot \underset{\sim}{\mu})}$$

$$+ \underset{\sim}{d} \times \hat{\underset{\sim}{l}} = \underset{\sim}{d} \times \underset{\sim}{d}_{tt} J_0$$

$$\overset{(*)\, p.207}{=} \cancel{\underset{\sim}{d} \times (\nabla \cdot \underset{\sim}{\mu})} - \underset{\sim}{d} \times \underset{\sim}{\sigma} + \underset{\sim}{d} \times \hat{\underset{\sim}{l}}$$

$$\Rightarrow \boxed{\underset{\sim}{r}_{,\alpha} \times \underset{\sim}{N}_\alpha + \underset{\sim}{d}_{,\alpha} \times \underset{\sim}{\mu}_\alpha + \underset{\sim}{d} \times \underset{\sim}{\sigma} = \underset{\sim}{0}},$$

which should be viewed as a constitutive identity (like the symmetry of the stress tensor).

<u>Remark</u> It turns out that (AMB) p. is enough for the special Cosserat theory $|\underset{\sim}{d}| \equiv 1$ (which we impose later), i.e., (*) p. 207 is not really required. Yet (*) p. 207 together with the symmetry identity above is revealing from the point of view of constitutive theory, which we explore next.

<u>Corollary</u> Given $\underset{\sim}{M}_\alpha = \underset{\sim}{d} \times \underset{\sim}{\mu}_\alpha$ and $\hat{\underset{\sim}{g}} = \underset{\sim}{d} \times \hat{\underset{\sim}{l}}$, we immediately conclude that

$$\underset{\sim}{d} \cdot \underset{\sim}{M}_\alpha = \underset{\sim}{d} \cdot \hat{\underset{\sim}{g}} \equiv \underset{\sim}{0}$$

Hyperelasticity

Returning to p.191, recall that

$$\underset{=}{F}_c = \nabla \underset{\sim}{r} + I_3 \nabla \underset{\sim}{d} + \underset{\sim}{d} \otimes \underset{\sim}{e}_3 ,$$

where $\nabla \underset{\sim}{r} = \frac{\partial \underset{\sim}{r}}{\partial I_\alpha} \otimes \underset{\sim}{e}_\alpha$, $\nabla \underset{\sim}{d} = \frac{\partial \underset{\sim}{d}}{\partial I_\alpha} \otimes \underset{\sim}{e}_\alpha$

(see (*) p.190).

As in rod theory, we can motivate the form of the ~~stored~~ energy function via the constrained 3-d viewpoint:

$$(*) \quad \Phi\left(\underset{\sim}{r}_{,\alpha}, \underset{\sim}{d}, \underset{\sim}{d}_{,\alpha}\right) \quad (\alpha = 1,2)$$

$$\equiv \int_{-h/2}^{h/2} W\left(\underset{=}{F}_c\right) dI_3$$

It is instructive to "derive" the total potential energy expression for the plate Ω.

$$V = \int_\Gamma \int_{-h/2}^{h/2} W\left(\underset{=}{F}_c\right) dI$$

$$- \int_\Gamma \int_{-h/2}^{h/2} \hat{\underset{\sim}{b}} \cdot \underset{\sim}{f}_c \, dI_3 \, dA$$

$$- \int_{\Omega_T \cup \Omega_B} \underset{\approx}{S} \underset{\sim}{m} \cdot \underset{\sim}{\hat{f}}_C \; dA \qquad \mathcal{B} = \Omega \times (-h/2, h/2)$$

$$\Omega_T = \Omega \times \{h/2\}$$
$$\Omega_B = \Omega \times \{-h/2\}$$

Exercise 26 Carry out this calculation above and show ~~that~~ for the plate

$$\mathcal{V}[\underset{\sim}{r}, \underset{\sim}{d}] = \int_\Omega \Phi(\underset{\sim}{r}_{,\alpha}, \underset{\sim}{d}, \underset{\sim}{d}_{,\alpha})\, dA$$

$$- \int_\Omega \underset{\sim}{\hat{b}} \cdot \underset{\sim}{r}\, dA - \int_\Omega \underset{\sim}{\hat{\ell}} \cdot \underset{\sim}{d}\, dA$$

"dead"

It is instructive to take the 1st variation: For simplicity, let's assume that $\underset{\sim}{r}|_{\partial\Omega}$ and $\underset{\sim}{d}|_{\partial\Omega}$ are <u>specified</u>:

$$\delta\mathcal{V} = \frac{d}{d\varepsilon} \mathcal{V}[\underset{\sim}{r} + \varepsilon \underset{\sim}{\eta}, \underset{\sim}{d} + \varepsilon \underset{\sim}{\xi}]\Big|_{\varepsilon=0} = 0$$

$\forall$ sufficiently smooth $\underset{\sim}{\eta}, \underset{\sim}{\xi}$ such that $\underset{\sim}{\eta}|_{\partial\Omega} = \underset{\sim}{\xi}|_{\partial\Omega} = \underset{\sim}{0}$ (admissible).

$$\therefore \quad \delta V = \frac{d}{d\varepsilon}\left[\int_\Omega \Phi\left(\underset{\sim}{r}_{,\alpha} + \varepsilon \underset{\sim}{\eta}_{,\alpha}, \underset{\sim}{d} + \varepsilon \underset{\sim}{\xi}, \underset{\sim}{d}_{,\alpha} + \varepsilon \underset{\sim}{\xi}_{,\alpha}\right) dA \right.$$

$$\left. - \int_\Omega \left[\hat{\underset{\sim}{b}} \cdot (\underset{\sim}{r} + \varepsilon \underset{\sim}{\eta}) - \hat{\underset{\sim}{\ell}} \cdot (\underset{\sim}{d} + \varepsilon \underset{\sim}{\xi})\right] dA \right]\Bigg|_{\varepsilon=0}$$

$$= \int_\Omega \left[\left(\frac{\partial \Phi}{\partial \underset{\sim}{r}_{,\alpha}}(\;) \cdot \underset{\sim}{\eta}_{,\alpha} + \frac{\partial \Phi}{\partial \underset{\sim}{d}} \cdot \underset{\sim}{\xi} + \frac{\partial \Phi}{\partial \underset{\sim}{d}_{,\alpha}} \cdot \underset{\sim}{\xi}_{,\alpha}\right)\right.$$

$$\left. - \hat{\underset{\sim}{b}} \cdot \underset{\sim}{\eta} - \hat{\underset{\sim}{\ell}} \cdot \underset{\sim}{\xi}\right] dA = \underset{\sim}{0} \quad \forall \text{ adm } \underset{\sim}{\eta}, \underset{\sim}{\xi}$$

Integrate by parts:

$$\underset{\sim}{U} = \frac{\partial \Phi}{\partial \underset{\sim}{r}_{,\alpha}} \qquad dV = \underset{\sim}{\eta}_{,\alpha}\, dX_\alpha$$

$$d\underset{\sim}{U} = \frac{\partial}{\partial X_\alpha}\left(\frac{\partial \Phi}{\partial \underset{\sim}{r}_{,\alpha}}\right) dX_\alpha \quad \underline{V} = \underset{\sim}{\eta}$$

$$\delta V = \int_\Omega \left\{ -\left[\frac{\partial}{\partial X_\alpha}\left(\frac{\partial \Phi}{\partial \underset{\sim}{r}_{,\alpha}}\right) \cdot \underset{\sim}{\eta} + \frac{\partial}{\partial X_\alpha}\left(\frac{\partial \Phi}{\partial \underset{\sim}{d}_{,\alpha}}\right) \cdot \underset{\sim}{\xi}\right] \right.$$

$$\left. + \frac{\partial \Phi}{\partial \underset{\sim}{d}} \cdot \underset{\sim}{\xi} - \hat{\underset{\sim}{b}} \cdot \underset{\sim}{\eta} - \hat{\underset{\sim}{\ell}} \cdot \underset{\sim}{\xi} \right\} dA = 0$$

$$\forall \text{ adm } \underset{\sim}{\eta}, \underset{\sim}{\xi}$$

Euler-Langrange eq'ns: