Lecture 26

$$\frac{\partial}{\partial X_\alpha}\left(\frac{\partial \Phi}{\partial \underset{\sim}{r}_{,\alpha}}\right) + \hat{\underset{\sim}{b}} = \underset{\sim}{0}$$

$$\frac{\partial}{\partial X_\alpha}\left(\frac{\partial \Phi}{\partial \underset{\sim}{d}_{,\alpha}}\right) - \frac{\partial \Phi}{\partial \underset{\sim}{d}} + \hat{\underset{\sim}{l}} = \underset{\sim}{0}$$

These are static equil. equations - comparing with the static versions of (*) p. 198 and (*) p. 207, respectively, we deduce:

$$(*) \quad \boxed{\underset{\sim}{N}_\alpha = \frac{\partial \Phi}{\partial \underset{\sim}{r}_{,\alpha}}, \quad \underset{\sim}{\sigma} = \frac{\partial \Phi}{\partial \underset{\sim}{d}}, \quad \underset{\sim}{M}_\alpha = \frac{\partial \Phi}{\partial \underset{\sim}{d}_{,\alpha}}}$$

In anticipation of material-symmetry considerations, we can give (*) a more elegant tensorial form as follows:

A more accurate version of (*) p. 209 is

$$\Phi(\underset{\sim}{r}_{,\alpha}, \underset{\sim}{d}, \underset{\sim}{d}_{,\alpha}) = \Psi(\nabla \underset{\sim}{r}, \underset{\sim}{d} \otimes \underset{\sim}{e}_3, \nabla \underset{\sim}{d})$$

(recalling $\nabla \underset{\sim}{r} = \underset{\sim}{r}_{,\alpha} \otimes \underset{\sim}{e}_\alpha$, $\nabla \underset{\sim}{d} = \underset{\sim}{d}_{,\alpha} \otimes \underset{\sim}{e}_\alpha$).

Define $\underset{\approx}{F} \equiv \nabla \underset{\sim}{r}$, $\underset{\approx}{G} = \nabla \underset{\sim}{d}$, $\underset{\approx}{D} = \underset{\sim}{d} \otimes \underset{\sim}{e}_3$.

Note $\underset{\approx}{F}, \underset{\approx}{G} \in L(\mathbb{E}^2, \mathbb{E}^3)$. Now consider

$$\frac{d}{d\varepsilon}\,\Phi\left(\underset{\sim}{r}_{,1}+\varepsilon\underset{\sim}{\eta},\ \underset{\sim}{r}_{,2},\ \underset{\sim}{d},\ \underset{\sim}{d}_{,\alpha}\right)\Big|_{\varepsilon=0}$$

$$=\frac{d}{d\varepsilon}\,\Psi\left((\underset{\sim}{r}_{,1}+\varepsilon\underset{\sim}{\eta})\otimes\underset{\sim}{e}_1+\underset{\sim}{r}_{,2},\ \underset{\sim}{d}\otimes\underset{\sim}{e}_3,\ \nabla\underset{\sim}{d}\right)\Big|_{\varepsilon=0}$$

$$\frac{\partial\Phi}{\partial\underset{\sim}{r}_{,1}}\cdot\underset{\sim}{\eta}=\frac{\partial\Psi}{\partial\underset{\approx}{F}}\cdot\underset{\sim}{\eta}\otimes\underset{\sim}{e}_1=\underset{\sim}{\eta}\cdot\frac{\partial\Psi}{\partial\underset{\approx}{F}}\underset{\sim}{e}_1$$

$$\forall\,\underset{\sim}{\eta}\in\mathbb{E}^3$$

(p. 212)

$$\Rightarrow\qquad\underset{\sim}{N}_1=\frac{\partial\Phi}{\partial\underset{\sim}{r}_{,1}}=\frac{\partial\Psi}{\partial\underset{\approx}{F}}\underset{\sim}{e}_1$$

p. 196

$$\Rightarrow\qquad\underset{\approx}{N}=\frac{\partial\Psi}{\partial\underset{\approx}{F}}$$

Similarly $\underset{\approx}{M}=\dfrac{\partial\Psi}{\partial\underset{\approx}{G}}$, and

$$\underset{\sim}{\sigma}=\frac{\partial\Phi}{\partial\underset{\sim}{d}}=\frac{\partial\Psi}{\partial\underset{\approx}{D}}\underset{\sim}{e}_3\,.$$

Material Objectivity

Borrowing again from the 3-d ~~theory~~ and (*) p. 209, we require the stored energy function to satisfy rotational invariance:

$$\text{(MO)}\quad\Phi\left(\underset{\approx}{Q}\underset{\sim}{r}_{,\alpha},\ \underset{\approx}{Q}\underset{\sim}{d},\ \underset{\approx}{Q}\underset{\sim}{d}_{,\alpha}\right)=\Phi\left(\underset{\sim}{r}_{,\alpha},\ \underset{\sim}{d},\ \underset{\sim}{d}_{,\alpha}\right)$$

$$\forall\,\underset{\approx}{Q}\in SO(3).$$

We now pursue a result analogously to that on p. 44:

__Thm__ (MO) p. 213 + hyperelasticity p. 212 $\Rightarrow$ AMB p. 208 (presuming LMB and director momentum balance).
* p. 207

__Pf__ As before, choose $\hat{\mathbf{Q}}(t) = \exp(\mathbf{\Omega} t)$ with $\mathbf{\Omega} \in \operatorname{skew}(\mathbb{E}^3)$. As shown on pp. 44-45, $\hat{\mathbf{Q}}(t) \in SO(3)$ for all t. Substituting $\hat{\mathbf{Q}}(t)$ into (MO) and differentiation w.r.t. t yields (using (*) p. 212):

$$\left[\mathbf{N}_\alpha \cdot \mathbf{\Omega}\, \cancelto{\mathbf{I}}{\exp(\mathbf{\Omega} t)}\, \mathbf{r}_{,\alpha} + \boldsymbol{\sigma} \cdot \mathbf{\Omega}\, \cancelto{\mathbf{I}}{\exp(\mathbf{\Omega} t)}\, \mathbf{d} + \mathbf{M}_\alpha \cdot \mathbf{\Omega}\, \cancelto{\mathbf{I}}{\exp(\mathbf{\Omega} t)}\, \mathbf{d}_{,\alpha}\right]\Bigg|_{t=0} = 0 \quad \forall\, \mathbf{\Omega}$$

$$\Rightarrow \quad \mathbf{N}_\alpha \cdot (\boldsymbol{\omega} \times \mathbf{r}_{,\alpha}) + \boldsymbol{\sigma} \cdot (\boldsymbol{\omega} \times \mathbf{d}) + \mathbf{M}_\alpha \cdot (\boldsymbol{\omega} \times \mathbf{d}_{,\alpha}) = 0$$

$$\forall\, \boldsymbol{\omega} = \operatorname{axial}(\mathbf{\Omega}) \in \mathbb{E}^3$$

Now $\mathbf{N}_\alpha \cdot (\boldsymbol{\omega} \times \mathbf{r}_{,\alpha}) = \mathbf{r}_{,\alpha} \cdot (\mathbf{N}_\alpha \times \boldsymbol{\omega})$

$$= \boldsymbol{\omega} \cdot (\mathbf{r}_{,\alpha} \times \mathbf{N}_\alpha), \text{ etc.}$$

$$\Rightarrow \quad \boldsymbol{\omega} \cdot (\mathbf{r}_{,\alpha} \times \mathbf{N}_\alpha + \mathbf{d} \times \boldsymbol{\sigma} + \mathbf{d}_{,\alpha} \times \mathbf{M}_\alpha) = 0 \quad \forall\, \boldsymbol{\omega} \in \mathbb{E}^3$$

$$\Rightarrow \quad \mathbf{r}_{,\alpha} \times \mathbf{N}_\alpha + \mathbf{d} \times \boldsymbol{\sigma} + \mathbf{d}_{,\alpha} \times \mathbf{M}_\alpha = \mathbf{0}. \quad \square$$

To exploit (MO) we write

$$\underline{\underline{F}}_m = \nabla \underset{\sim}{r} + \underset{\sim}{d} \otimes \underset{\sim}{e}_3 \equiv \underline{\underline{F}}_c \Big|_{X_3 = 0}$$

mid-surface
deformation gradient,

and $\Psi(\underline{\underline{F}}_m, \underline{\underline{G}})$.

(MO) reads

$$(*) \quad \Psi(\underline{\underline{Q}}\,\underline{\underline{F}}_m, \underline{\underline{Q}}\,\underline{\underline{G}}) = \Psi(\underline{\underline{F}}_m, \underline{\underline{G}})$$

$$\forall\, \underline{\underline{Q}} \in SO(3).$$

<u>Claim</u>: (MO) (*) (above) $\Rightarrow$

$$\Psi(\underline{\underline{F}}_m, \underline{\underline{G}}) = \Upsilon(\underline{\underline{F}}_m^T \underline{\underline{F}}_m, \underline{\underline{F}}_m^T \underline{\underline{G}})$$

$$\equiv \Upsilon(\underline{\underline{C}}_m, \underline{\underline{F}}_m^T \underline{\underline{G}})$$

<u>Pf</u> As before (p. 41) write $\underline{\underline{F}}_m = \underline{\underline{R}}_m \underline{\underline{U}}_m$

and choose $\underline{\underline{Q}} = \underline{\underline{R}}_m^T$ in (*) above:

$$\Rightarrow \Psi(\underline{\underline{F}}_m, \underline{\underline{G}}) = \Psi(\underline{\underline{U}}_m, \underline{\underline{R}}_m^T \underline{\underline{G}})$$

$$\equiv \tilde{\Psi}(\underline{\underline{U}}_m^2, \underline{\underline{U}}_m \underline{\underline{R}}_m^T \underline{\underline{G}})$$

$$= \Upsilon(\underline{\underline{C}}_m, \underline{\underline{F}}_m^T \underline{\underline{G}}). \quad \square$$

Strain Measures

$$\underline{\underline{F}}_m^T \underline{\underline{F}}_m = (\underline{e}_3 \otimes \underline{d} + \underline{\nabla}\underline{r}^T)(\underline{\nabla}\underline{r} + \underline{d} \otimes \underline{e}_3)$$

$$= \underline{\nabla}\underline{r}^T \underline{\nabla}\underline{r} + \underline{\nabla}\underline{r}^T \underline{d} \otimes \underline{e}_3 + \underline{e}_3 \otimes \underline{d}\, \underline{\nabla}\underline{r}$$

$$+ \ \underline{e}_3 \otimes \underline{d}\, (\underline{d} \otimes \underline{e}_3)$$

Observe $\underline{\nabla}\underline{r}^T \in L(\mathbb{E}^3, \mathbb{E}^2)$

$$\Rightarrow \underline{\nabla}\underline{r}^T \underline{\nabla}\underline{r} \in L(\mathbb{E}^2, \mathbb{E}^2)$$

Indeed, $\underline{\nabla}\underline{r} = \underline{r}_{,\alpha} \otimes \underline{e}_\alpha$

$$\underline{\nabla}\underline{r}^T \underline{\nabla}\underline{r} = \underline{e}_\alpha \otimes \underline{r}_{,\alpha}\, \underline{r}_{,\beta} \otimes \underline{e}_\beta$$

$$= \underline{r}_{,\alpha} \cdot \underline{r}_{,\beta}\ \underline{e}_\alpha \otimes \underline{e}_\beta$$

$$= a_{\alpha\beta}\ \underline{e}_\alpha \otimes \underline{e}_\beta \equiv \underline{\underline{C}}$$ "in-plane right Cauchy-Green"

components of "1st fund. form" in differential geometry.

$$\underline{\nabla}\underline{r}^T \underline{d} \otimes \underline{e}_3 = \underline{e}_\alpha \otimes \underline{r}_{,\alpha}\, \underline{d} \otimes \underline{e}_3$$

$$= \underline{d} \cdot \underline{r}_{,\alpha}\ \underline{e}_\alpha \otimes \underline{e}_3 \equiv \gamma_\alpha\, \underline{e}_\alpha \otimes \underline{e}_3$$

$$\underline{e}_3 \otimes \underline{d}\, \underline{\nabla}\underline{r} = \underline{e}_3 \otimes \underline{d}\ \underline{r}_{,\alpha} \otimes \underline{e}_\alpha = \gamma_\alpha\, \underline{e}_3 \otimes \underline{e}_\alpha$$

$$\mathbf{e}_3 \otimes \mathbf{d}\,(\mathbf{d} \otimes \mathbf{e}_3) = |\mathbf{d}|^2\, \mathbf{e}_3 \otimes \mathbf{e}_3$$

$$= \lambda^2\, \mathbf{e}_3 \otimes \mathbf{e}_3$$

Also,

$$\mathbf{F}_m^T \mathbf{G} = (\mathbf{e}_3 \otimes \mathbf{d} + \nabla \mathbf{r}^T)(\mathbf{d}_{,\alpha} \otimes \mathbf{e}_\alpha)$$

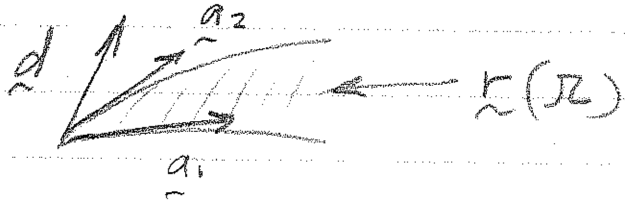
$$= \mathbf{d} \cdot \mathbf{d}_{,\alpha}\, \mathbf{e}_3 \otimes \mathbf{e}_\alpha + \mathbf{e}_\beta \otimes \mathbf{r}_{,\beta}\, \mathbf{d}_{,\alpha} \otimes \mathbf{e}_\alpha$$

$$\mathbf{r}_{,\beta} \cdot \mathbf{d}_{,\alpha}\, \mathbf{e}_\beta \otimes \mathbf{e}_\alpha$$

$$= \delta_\alpha\, \mathbf{e}_3 \otimes \mathbf{e}_\alpha + \kappa_{\alpha\beta}\, \mathbf{e}_\alpha \otimes \mathbf{e}_\beta$$

$$\delta_\alpha \equiv \mathbf{d} \cdot \mathbf{d}_{,\alpha}, \quad \kappa_{\alpha\beta} \equiv \mathbf{r}_{,\alpha} \cdot \mathbf{d}_{,\beta}$$

Note that $\mathbf{a}_\alpha \equiv \mathbf{r}_{,\alpha}$ are tangent vectors on the surface $\mathbf{r}(\Omega)$



Clearly $\gamma_\alpha \equiv \mathbf{a}_\alpha \cdot \mathbf{d}$ are nonzero when $\mathbf{d}$ is <u>not</u> $\perp$ to the tangent plane span$\{\mathbf{a}_1, \mathbf{a}_2\}$, and we call γ_α the <u>transverse shears</u>.