Lecture 27

The other "strains" are less transparent:

$K_{\alpha\beta} = \underset{\sim}{a}_\alpha \cdot \underset{\sim}{d}_{,\beta}$ is related to the 2^{nd} fundamental form of the surface $\underset{\sim}{r}(\Omega)$. If $\underset{\sim}{d} \equiv \underset{\sim}{n}$ unit normal, then "$-K_{\alpha\beta}$" are precisely the components of the 2^{nd} fund. form.

"Kirchhoff-Love" theory

Finally $\delta_\alpha = \underset{\sim}{d} \cdot \underset{\sim}{d}_{,\alpha}$, $\alpha = 1, 2$, are sometimes called the "couple shears".

Let's now focus our attention on the so-called special Cosserat theory, viz., we make the assumption

$$(*) \quad \underset{\sim}{d} \cdot \underset{\sim}{d} = 1 \quad \text{(unit director)}$$

From the top of p. 217, we conclude that

$$\boxed{\lambda = 1,}$$

Similarly, we have

$$(\underset{\sim}{d} \cdot \underset{\sim}{d})_{,\alpha} = 2\underset{\sim}{d} \cdot \underset{\sim}{d}_{,\alpha} = 0$$

$$\Rightarrow \quad \boxed{\delta_\alpha = 0, \quad \alpha = 1, 2.}$$

Hence the remaining strain measures are

$\underset{\approx}{e} = a_{\alpha\beta}\, \underset{\sim}{e}_\alpha \otimes \underset{\sim}{e}_\beta$, $\underset{\approx}{K} = K_{\alpha\beta}\, \underset{\sim}{e}_\alpha \otimes \underset{\sim}{e}_\beta$ and $\underset{\sim}{\delta} = \delta_\alpha \underset{\sim}{e}_\alpha$, Thus the free-energy density reduces to

$$\underline{\Phi} = \Psi\left(\underline{\underline{C}}, \underline{\underline{K}}, \underset{\sim}{\gamma}\right)$$

Now in this constrained ~~theory~~ ((*) p. 218) it is illuminating to observe that we essentially have 5 unknown fields — $\underset{\sim}{r}$, $\underset{\sim}{d}$ subject to $|\underset{\sim}{d}| = 1$ — and then to also consider $\underset{\sim}{d} \times$ (*) p. 207:

$$\underbrace{\underset{\sim}{d} \times \big(\underset{\sim}{\mu}_{\alpha,\alpha}}{} - \underset{\sim}{\sigma} + \hat{\underset{\sim}{\ell}}\big) = J_0\, \underset{\sim}{d} \times \underset{\sim}{d}_{tt}$$

$$\Rightarrow \left(\underset{\sim}{d} \times \underset{\sim}{\mu}_{\alpha}\right)_{,\alpha} - \underset{\sim}{d}_{,\alpha} \times \underset{\sim}{\mu}_{\alpha}$$

$$- \underset{\sim}{d} \times \underset{\sim}{\sigma} + \underset{\sim}{d} \times \hat{\underset{\sim}{\ell}} = J_0\, \underset{\sim}{d} \times \underset{\sim}{d}_{tt}$$

p. 204

$$\Rightarrow \underbrace{\underset{\sim}{M}_{\alpha,\alpha}}_{\nabla \cdot \underline{\underline{M}}} - \left(\underset{\sim}{d}_{,\alpha} \times \underset{\sim}{\mu}_{\alpha} + \underset{\sim}{d} \times \underset{\sim}{\sigma}\right) + \hat{\underset{\sim}{g}} = J_0\, \underset{\sim}{d} \times \underset{\sim}{d}_{tt}$$

But by virtue of the Thm. on p. 214, we have (hyperelasticity)

$$-\left(\underset{\sim}{d}_{,\alpha} \times \underset{\sim}{\mu}_{\alpha} + \underset{\sim}{d} \times \underset{\sim}{\sigma}\right) = \underset{\sim}{r}_{,\alpha} \times \underset{\sim}{N}_{\alpha}$$

$$\Rightarrow \underset{\sim}{M}_{\alpha,\alpha} + \underset{\sim}{r}_{,\alpha} \times \underset{\sim}{N}_{\alpha} + \hat{\underset{\sim}{g}} = J_0\, \underset{\sim}{d} \times \underset{\sim}{d}_{tt}$$

Of course this is simply (AMB) again (cf. (*) p. 203)

However, what we learn here is that

$\underset{\sim}{d} \cdot (AMB) = 0$, i.e., (AMB) essentially delivers 2 scalar equations. So (LMB) + (AMB) gives essentially 5 equations, which are enough (with constitutive laws) to determine the 5 unknowns $\underset{\sim}{r}$, $\underset{\sim}{d}$ (subject to $|\underset{\sim}{d}| \equiv 1$). In the absence of the constraint, we need (LMB) + (director MB) (p. 207), the latter of which has 3 components.

Henceforth, we concentrate solely on <u>Kirchhoff plates</u>:

We enforce not only $(*)\ |\underset{\sim}{d}| \equiv 1$, but also the two constraints

$$(**)\ \underset{\sim}{a}_\alpha \cdot \underset{\sim}{d} \equiv 0 \quad , \alpha = 1, 2 .$$

Together $(*)$ & $(**)$ $\Rightarrow$ $\underset{\sim}{d}$ is a <u>unit normal</u> vector field, i.e.,

$$\underset{\sim}{d} = \frac{\underset{\sim}{a}_1 \times \underset{\sim}{a}_2}{|\underset{\sim}{a}_1 \times \underset{\sim}{a}_2|} \equiv \underset{\sim}{n}$$

Exercise 27 Show that

$$J \equiv |\underset{\sim}{a}_1 \times \underset{\sim}{a}_2| = \sqrt{\det \underset{\sim}{C}} \quad \text{local area change.}$$

<u>Hint</u> $\underset{\sim}{F}_m \overset{(p.215)}{=} \underset{\sim}{a}_\alpha \otimes \underset{\sim}{e}_\alpha + \underset{\sim}{n} \otimes \underset{\sim}{e}_3$;

compute $\det \underset{\approx}{C}_m$.

Of course $(**)$ p. 220 shows that the shear strains γ_α, $\alpha = 1,2$, are zero. But also

$$K_{\alpha\beta} \overset{(p.217)}{=} \underset{\sim}{a}_\alpha \cdot \underset{\sim}{d}_{,\beta} = \underset{\sim}{a}_\alpha \cdot \underset{\sim}{n}_{,\beta} = -b_{\alpha\beta}$$

(components of the 2^{nd} fund. form in diff. geom)

Observe $\quad \underset{\sim}{a}_\alpha \cdot \underset{\sim}{n} \equiv 0$

$$\Rightarrow \underset{\sim}{a}_{\alpha,\beta} \cdot \underset{\sim}{n} + \underset{\sim}{a}_\alpha \cdot \underset{\sim}{n}_{,\beta} = 0$$

or
$$K_{\alpha\beta} = -\underset{\sim}{a}_{\alpha,\beta} \cdot \underset{\sim}{n}$$

$$= -\underset{\sim}{a}_{\beta,\alpha} \cdot \underset{\sim}{n} = K_{\beta\alpha},$$

symmetric!

i.e., $\underset{\approx}{\mathcal{K}} = K_{\alpha\beta}\, \underset{\sim}{e}_\alpha \otimes \underset{\sim}{e}_\beta$,

where $\underset{\approx}{\mathcal{K}} \equiv \nabla \underset{\sim}{r}^T \nabla \underset{\sim}{d}$ (p. 217), then

$$\underset{\approx}{\mathcal{K}}^T = \underset{\approx}{\mathcal{K}}.$$

Next let's proceed with the structure of the field equations in the presence of $(*)$ & $(**)$ p. 220 via Lagrange multipliers:

From p. 210, we have

$$\mathcal{V} = \int_\Omega \Big[\underbrace{\Phi(\underset{\sim}{r}_{,\alpha}, \underset{\sim}{d}, \underset{\sim}{d}_{,\alpha})}_{(**)} - \overbrace{\tfrac{\mu}{2}(\underset{\sim}{d}\cdot\underset{\sim}{d}-1)}^{(*)}$$

$$- q_\alpha\, \underset{\sim}{r}_{,\alpha}\cdot\underset{\sim}{d} \Big]\, dA + \ldots$$

Then

(p. 210)

$$\delta\mathcal{V} = \int_\Omega \underbrace{\left(\frac{\partial\Phi}{\partial\underset{\sim}{r}_{,\alpha}} - q_\alpha \underset{\sim}{d}\right)}_{\sim \underset{\sim}{N}_\alpha}\cdot\underset{\sim}{\eta}_{,\alpha}$$

$$+ \underbrace{\left[\frac{\partial\Phi}{\partial\underset{\sim}{d}} - \left(\mu\underset{\sim}{d} + q_\alpha\underset{\sim}{a}_\alpha\right)\right]}_{\sim \underset{\sim}{\sigma}}\cdot\underset{\sim}{\zeta}$$

$$+ \ldots = 0 \quad \forall \text{ adm } \underset{\sim}{\eta}, \underset{\sim}{\zeta}$$

The first term above shows that $\underset{\sim}{N}_\alpha$, $\alpha = 1,2$, has a constitutively indeterminate component along $\underset{\sim}{d}$:

$$(*)\quad \boxed{\begin{aligned} &\underset{\sim}{N}_\alpha = \bar{\underset{\sim}{N}}_\alpha + q_\alpha \underset{\sim}{d} \\ &\bar{\underset{\sim}{N}}_\alpha = \frac{\partial\Phi}{\partial\underset{\sim}{r}_{,\alpha}}, \quad \bar{\underset{\sim}{N}}_\alpha\cdot\underset{\sim}{d} \equiv 0,\ \alpha=1,2. \end{aligned}}$$

The second expression shows that

$\underset{\sim}{\sigma} = q_\alpha \underset{\sim}{a}_\alpha + \mu\underset{\sim}{d}$ is constitutively indeterminate which, as shown on p.219, falls out of (AMB).

In-Plane Material Symmetry

Define the symmetry operations

$$\underset{\approx}{R}(\theta) = \cos\theta\,(\underset{\sim}{e}_1\otimes\underset{\sim}{e}_1 + \underset{\sim}{e}_2\otimes\underset{\sim}{e}_2) + \sin\theta\,(\underset{\sim}{e}_1\otimes\underset{\sim}{e}_2 - \underset{\sim}{e}_2\otimes\underset{\sim}{e}_1),$$
$$\underset{\approx}{\Xi} = \underset{\sim}{e}_1\otimes\underset{\sim}{e}_1 - \underset{\sim}{e}_2\otimes\underset{\sim}{e}_2,$$

and the orthogonal group on E^2:

$$O(2) = \{\underset{\approx}{R}(\theta),\ \underset{\approx}{\Xi}\underset{\approx}{R}(\theta) : 0\le\theta<2\pi\}.$$

We take the same approach as before to motivate our definition via two "armchair experiments:

① Apply a deformation such that the values such that the strains at $\underset{\sim}{s}\in\Omega$ are $\underset{\approx}{C}$ and $\underset{\approx}{K}$ - say coming from

$$\underset{\approx}{F}_c^{①} = \underset{\sim}{a}_\alpha\otimes\underset{\sim}{e}_\alpha + \underset{\sim}{d}\otimes\underset{\sim}{e}_3 + X_3\,\underset{\sim}{d}_{,\alpha}\otimes\underset{\sim}{e}_\alpha.$$
$$= \underset{\approx}{F} + \underset{\sim}{d}\otimes\underset{\sim}{e}_3 + X_3\,\underset{\approx}{G}\quad,\ \text{where}$$

$$\underset{\approx}{C} = \underset{\approx}{F}^T\underset{\approx}{F}\ ,\quad \underset{\approx}{K} = \underset{\approx}{F}^T\underset{\approx}{G}.$$

We then "measure" $\mathcal{N}(\underset{\approx}{C},\underset{\approx}{K})$.

(2) First rotate or reflect Ω according to $\underline{\underline{Q}} \in O(2)$, and then perform $\underline{f}_c^{(1)}$ on the transformed body $\underline{\underline{Q}}(\Omega)$:

$$\underline{\underline{F}}_c^{(2)} = \underline{d}_\alpha \otimes (\underline{\underline{Q}}\underline{e}_\alpha) + \underline{d} \otimes \underline{e}_3 + X_3\, \underline{d}_{,\alpha} \otimes (\underline{\underline{Q}}\underline{e}_\alpha)$$

$$= \underline{\underline{F}}\,\underline{\underline{Q}}^T + \underline{d} \otimes \underline{e}_3 + X_3\, \underline{\underline{G}}\,\underline{\underline{Q}}^T,$$

i.e., $\underline{\underline{F}} \to \underline{\underline{F}}\,\underline{\underline{Q}}^T$, $\underline{\underline{G}} \to \underline{\underline{G}}\,\underline{\underline{Q}}^T$, and thus,

$$\underline{\underline{C}} \to (\underline{\underline{F}}\,\underline{\underline{Q}}^T)^T \underline{\underline{F}}\,\underline{\underline{Q}}^T = \underline{\underline{Q}}\,\underline{\underline{C}}\,\underline{\underline{Q}}^T$$

$$\underline{\underline{K}} \to (\underline{\underline{F}}\,\underline{\underline{Q}}^T)^T \underline{\underline{F}}\,\underline{\underline{Q}}^T = \underline{\underline{Q}}\,\underline{\underline{K}}\,\underline{\underline{Q}}^T.$$

Again, we "measure" $\Upsilon(\underline{\underline{Q}}\,\underline{\underline{C}}\,\underline{\underline{Q}}^T, \underline{\underline{Q}}\,\underline{\underline{K}}\,\underline{\underline{Q}}^T)$.

<u>Definition</u>: $\underline{\underline{Q}} \in O(2)$ is an in-plane material symmetry (of Ω at $\underline{s}$) if

$$(*) \quad \Upsilon(\underline{\underline{Q}}\,\underline{\underline{C}}\,\underline{\underline{Q}}^T, \underline{\underline{Q}}\,\underline{\underline{K}}\,\underline{\underline{Q}}^T) = \Upsilon(\underline{\underline{C}}, \underline{\underline{K}})$$

for all $\underline{\underline{C}}, \underline{\underline{K}} \in Sym(E^2)$
(actually $\underline{\underline{C}} \in Sym^+(E^2)$).

If $(*)$ holds for all $\underline{\underline{Q}} \in O(2)$, we say that Ω is <u>in-plane</u> <u>isotropic</u>.