Lecture 28

Thm Ω in-plane isotropic $\Longleftrightarrow$

$$\Omega = \Phi(I_1, I_2, \ldots, I_5),$$

where $I_1 = \operatorname{tr} \underline{\underline{C}}$, $I_2 = \det \underline{\underline{C}}$, $I_3 = \operatorname{tr} \underline{\underline{K}}$, $I_4 = \det \underline{\underline{K}}$, $I_5 = \underline{\underline{C}} \cdot \underline{\underline{K}}$.

Pf "$\Leftarrow$" is easy. For "$\Rightarrow$", we demonstrate that

$$\Omega(\underline{\underline{C}}^*, \underline{\underline{K}}^*) = \Omega(\underline{\underline{C}}, \underline{\underline{K}})$$

whenever $(\underline{\underline{C}}^*, \underline{\underline{K}}^*)$ and $(\underline{\underline{C}}, \underline{\underline{K}})$ have the same invariants $I_1, I_2, \ldots, I_5$. To see this, observe that $\underline{\underline{C}}^*$ and $\underline{\underline{C}}$ have the same invariants, I_1 and I_2, and $\underline{\underline{K}}^*$ and $\underline{\underline{K}}$ have the same invariants, I_3 and I_4. Recall that for a given $\underline{\underline{a}} \in \mathcal{L}(E^2)$, we have

$$\det(\underline{\underline{a}} - \lambda \underline{\underline{I}}) = \lambda^2 - \operatorname{tr} \underline{\underline{a}}\, \lambda + \det \underline{\underline{a}}.$$

Thus, $\underline{\underline{C}}$ and $\underline{\underline{C}}^*$ each have the same characteristic equation and likewise $\underline{\underline{K}}$ and $\underline{\underline{K}}^*$. Since each of these is symmetric, we conclude (as on p. 53) that there are $\underline{\underline{Q}}_1, \underline{\underline{Q}}_2 \in O(2)$ such that

$$\underline{\underline{C}}^* = \underline{\underline{Q}}_1 \underline{\underline{C}}\, \underline{\underline{Q}}_1^T$$

and

$$\underline{\underline{K}}^* = \underline{\underline{Q}}_2 \underline{\underline{K}}\, \underline{\underline{Q}}_2^T$$

Next we use that

$$I_5 = \underline{\underline{C}}^* \cdot \underline{\underline{K}}^* = \underline{\underline{C}} \cdot \underline{\underline{K}} \quad \text{(given)}$$

$$\Rightarrow (*) \quad \underbrace{\underline{\underline{Q}}_1 \underline{\underline{C}}\, \underline{\underline{Q}}_1^T \cdot \underline{\underline{Q}}_2 \underline{\underline{K}}\, \underline{\underline{Q}}_2^T}_{} = \underline{\underline{C}} \cdot \underline{\underline{K}}$$

$$\mathrm{tr}\left(\underline{\underline{Q}}_1 \underline{\underline{C}}\, \underline{\underline{Q}}_1^T (\underline{\underline{Q}}_2 \underline{\underline{K}}\, \underline{\underline{Q}}_2^T)^T\right)$$

$$= \mathrm{tr}\left((\underline{\underline{Q}}_1 \underline{\underline{C}}\, \underline{\underline{Q}}_1^T \underline{\underline{Q}}_2 \underline{\underline{K}})\underline{\underline{Q}}_2^T\right) \quad (\underline{\underline{K}} \in \mathrm{Sym}(E^2))$$

$$= \mathrm{tr}\left((\underline{\underline{Q}}_2^T \underline{\underline{Q}}_1 \underline{\underline{C}}\, \underline{\underline{Q}}_1^T \underline{\underline{Q}}_2)\underline{\underline{K}}\right)$$

$$= \underline{\underline{Q}}_2^T \underline{\underline{Q}}_1 \underline{\underline{C}} (\underline{\underline{Q}}_2^T \underline{\underline{Q}}_1)^T \cdot \underline{\underline{K}}$$

Note $\underline{\underline{R}} \equiv \underline{\underline{Q}}_2^T \underline{\underline{Q}}_1 \in O(2)$.

$$\therefore (*) \Rightarrow \underline{\underline{R}}\, \underline{\underline{C}}\, \underline{\underline{R}}^T \cdot \underline{\underline{K}} = \underline{\underline{C}} \cdot \underline{\underline{K}}$$

$$\forall\, \underline{\underline{C}}, \underline{\underline{K}} \in \mathrm{Sym}(E^2)$$

In particular, this holds for all $\underline{\underline{K}} \in \mathrm{Sym}(E^2)$

$$\Rightarrow \quad \underline{\underline{R}}\, \underline{\underline{C}}\, \underline{\underline{R}}^T = \underline{\underline{C}} \quad \forall\, \underline{\underline{C}} \in \mathrm{Sym}(E^2)$$

and for some particular $\underline{\underline{R}} \in O(2)$. Clearly, the only way this is possible for a general $\underline{\underline{C}}$ is $\underline{\underline{R}} = \underline{\underline{I}}$, i.e., $\underline{\underline{Q}}_2 = \underline{\underline{Q}}_1 \equiv \underline{\underline{Q}} \in O(2)$.

Thus, $\quad \Upsilon(\underline{\underline{C}}^*, \underline{\underline{K}}^*) = \Upsilon(\underline{\underline{Q}}\, \underline{\underline{C}}\, \underline{\underline{Q}}^T, \underline{\underline{Q}}\, \underline{\underline{K}}\, \underline{\underline{Q}}^T)$

$$= \Upsilon(\underline{\underline{C}}, \underline{\underline{K}}) \text{ by isotropy.} \quad \square$$

Before we proceed, we examine

Mid-Plane Symmetry

What we have in mind here is to perform a reflection across the mid-surface plane span$\{\mathbf{e}_1, \mathbf{e}_2\}$, and then repeat "experiment (1)" (p. 223) on the reflected body:

(2)' $\quad \mathbf{f}_c^{(2)'}(\mathbf{X}) = \mathbf{r}(\mathbf{s}) - X_3 \mathbf{d}(\mathbf{s}) \quad (\mathbf{s} \to \mathbf{s}, X_3 \to -X_3)$

$$\Rightarrow \mathbf{F}_c^{(2)'} = \mathbf{a}_\alpha \otimes \mathbf{e}_\alpha - \mathbf{d} \otimes \mathbf{e}_3 - X_3 \mathbf{d}_{,\alpha} \otimes \mathbf{e}_\alpha$$

$$= \mathbb{F} - \mathbf{d} \otimes \mathbf{e}_3 - X_3 \mathbb{G}$$

$$\Rightarrow \mathbb{F} \to \mathbb{F}, \quad \mathbb{G} \to -\mathbb{G}$$

$$\Rightarrow \mathbf{C} \to \mathbf{C}, \quad \mathbf{K} = \mathbb{F}^T\mathbb{G} \to -\mathbb{F}^T\mathbb{G} = -\mathbf{K}$$

Let's combine this with in-plane isotropy:

Clearly I_1, I_2 are unchanged.

$$I_3 = \operatorname{tr} \mathbf{K} \longrightarrow \operatorname{tr}(-\mathbf{K}) = -\operatorname{tr} \mathbf{K} = -I_3$$

$$I_4 = \det \mathbf{K} \longrightarrow \det(-\mathbf{K}) = \det(\mathbf{K}) = I_4$$

$$I_5 = \mathbf{C} \cdot \mathbf{K} \longrightarrow \mathbf{C} \cdot (-\mathbf{K}) = -\mathbf{C} \cdot \mathbf{K} = -I_5.$$

$\therefore$ In-plane isotropy + mid-plane symmetry:

$$(*) \quad \Upsilon(I_1, I_2, -I_3, I_4, -I_5) = \Upsilon(I_1, I_2, \ldots, I_5).$$

Thm $(*) \Leftrightarrow$

$$\Phi = \Upsilon = \Gamma(I_1, I_2, I_3^2, I_4, I_5^2, I_3 I_5).$$

Pf (see p. 161).

Stress and Stress-Couple Resultants (Kirchhoff Theory)

Returning to p. 212, in the context of the Kirchhoff theory ($\underset{\sim}{\sigma}$ is constitutively indeterminate (p. 222 - bottom)), we now write the stored-energy function as

$$\Phi = \Psi(\underset{\approx}{F}, \underset{\approx}{G})$$
$$= \Upsilon(\underset{\approx}{C}, \underset{\approx}{K}) - \text{using (MO) (p. 215)}.$$

Now from p. 213, we have

$$\underset{\approx}{N} = \frac{\partial \Psi}{\partial \underset{\approx}{F}} \quad \text{and} \quad \underset{\approx}{\mu} = \frac{\partial \Psi}{\partial \underset{\approx}{G}}$$

Our goal here is to work out what these