$\therefore$ In-plane isotropy + mid-plane symmetry:

(*) $\Upsilon(I_1, I_2, -I_3, I_4, -I_5) = \Upsilon(I_1, I_2, \dots, I_5)$.

Thm (*) $\Leftrightarrow$

$$\Phi = \Upsilon = \Gamma(I_1, I_2, I_3^2, I_4, I_5^2, I_3 I_5).$$

Pf (see p. 161).

Lecture 29

Stress and Stress-Couple Resultants (Kirchhoff Theory)

Returning to p. 212, in the context of the Kirchhoff ~~theory~~ ($\underset{\sim}{\sigma}$ is constitutively indeterminate (p. 222 - bottom)), we now write the stored-energy function as

$$\Phi = \Psi(\underset{=}{F}, \underset{=}{G})$$

$$= \Upsilon(\underset{=}{C}, \underset{=}{K}) \text{ - using (Mo) (p. 215).}$$

Now from p. 213, we have

$$\underset{\approx}{N} = \frac{\partial \Psi}{\partial \underset{=}{F}} \quad \text{and} \quad \underset{=}{\mu} = \frac{\partial \Psi}{\partial \underset{=}{G}}$$

Our goal here is to work out what these

look like in terms of $\Upsilon(\underline{\underline{C}}, \underline{\underline{K}})$. Consider

$$\frac{d}{d\alpha} \Psi(\underline{\underline{F}} + \alpha \underline{\underline{A}}, \underline{\underline{G}})\Big|_{\alpha=0} \qquad \underline{\underline{A}} \in L(E^3; E^3)$$

$$= \frac{d}{d\alpha} \Upsilon\left((\underline{\underline{F}}^T + \alpha \underline{\underline{A}}^T)(\underline{\underline{F}} + \alpha \underline{\underline{A}}), (\underline{\underline{F}}^T + \alpha \underline{\underline{A}}^T)\underline{\underline{G}}\right)\Big|_{\alpha=0}$$

$$\frac{\partial \Psi}{\partial \underline{\underline{F}}} \cdot \underline{\underline{A}} = \frac{\partial \Upsilon}{\partial \underline{\underline{C}}} \cdot (\underline{\underline{F}}^T \underline{\underline{A}} + \underline{\underline{A}}^T \underline{\underline{F}})$$

$$+ \frac{\partial \Upsilon}{\partial \underline{\underline{K}}} \cdot \underline{\underline{A}}^T \underline{\underline{G}}$$

Call $\underline{\underline{N}} = \frac{\partial \Upsilon}{\partial \underline{\underline{C}}}$ and $\underline{\underline{M}} \equiv \frac{\partial \Upsilon}{\partial \underline{\underline{K}}}$

Then $\underline{\underline{N}} \cdot (\underline{\underline{F}}^T \underline{\underline{A}} + \underline{\underline{A}}^T \underline{\underline{F}})$

$$= N_{\alpha\beta}\left[(\underline{\underline{F}}^T)_{\alpha i} A_{i\beta} + (\underline{\underline{A}}^T)_{\alpha i} F_{i\beta}\right]$$

$$= N_{\alpha\beta}(F_{i\alpha} A_{i\beta} + A_{i\alpha} F_{i\beta})$$

$$= F_{i\alpha} N_{\alpha\beta} A_{i\beta} + F_{i\beta} N_{\alpha\beta} A_{i\alpha}$$

$$= \underline{\underline{F}}\underline{\underline{N}} \cdot \underline{\underline{A}} + \underline{\underline{F}}\underline{\underline{N}}^T \cdot \underline{\underline{A}}$$

$$= 2\underline{\underline{F}}\underline{\underline{N}} \cdot \underline{\underline{A}} \qquad (\underline{\underline{N}}^T = \underline{\underline{N}} \text{ why?})$$

$$\therefore \ \mathbb{N} \cdot \mathbf{A} = \frac{\partial \Psi}{\partial \mathbf{F}} \cdot \mathbf{A}$$

$$= \left( 2\mathbf{F} \frac{\partial \Psi}{\partial \mathbf{C}} + \mathbf{G} \frac{\partial \Psi}{\partial \mathbf{K}} \right) \cdot \mathbf{A} \quad \forall \mathbf{A}$$

$$(*) \Rightarrow \boxed{\mathbb{N} = 2\mathbf{F} \frac{\partial \Psi}{\partial \mathbf{C}} + \mathbf{G} \frac{\partial \Psi}{\partial \mathbf{K}}}$$

<u>Observe</u> $\frac{\partial \Psi}{\partial \mathbf{C}} \mathbf{e}_\alpha , \ \frac{\partial \Psi}{\partial \mathbf{K}} \mathbf{e}_\alpha \in \mathbb{E}^2 = \text{span}\{\mathbf{e}_1, \mathbf{e}_2\}$

$$\therefore \ \mathbb{N}\mathbf{e}_\alpha = 2\mathbf{F}(\nu_\gamma \mathbf{e}_\gamma) + \mathbf{G}(\mu_\gamma \mathbf{e}_\gamma)$$

$$= 2(\mathbf{a}_\beta \otimes \mathbf{e}_\beta)(\nu_\gamma \mathbf{e}_\gamma) + (\mathbf{n}_{,\beta} \otimes \mathbf{e}_\beta)(\mu_\gamma \mathbf{e}_\gamma)$$

$$= 2\nu_\beta \mathbf{a}_\beta + \mu_\beta \mathbf{n}_{,\beta} ,$$

and $\mathbf{n} \cdot \mathbb{N}\mathbf{e}_\alpha = 2\nu_\beta \cancelto{0}{\mathbf{a}_\beta \cdot \mathbf{n}} + \mu_\beta \cancelto{0}{\mathbf{n}_{,\beta} \cdot \mathbf{n}}$

$(\mathbf{n} \cdot \mathbf{n} \equiv 1)$

$$\Rightarrow \bar{\mathbf{N}}_\alpha = \boxed{2\mathbf{F} \frac{\partial \Psi}{\partial \mathbf{C}} + \mathbf{G} \frac{\partial \Psi}{\partial \mathbf{K}}}$$

((*) p. 222)

$$\searrow \ \mathbf{N}_\alpha = \bar{\mathbf{N}}_\alpha + q_\alpha \mathbf{n}$$

Similarly,

$$\frac{d}{d\alpha}\Psi(\mathbf{F}, \mathbf{G}+\alpha\mathbf{B})\Big|_{\alpha=0} = \frac{d}{d\alpha}\Upsilon(\mathbf{C}, \mathbf{F}^T(\mathbf{G}+\alpha\mathbf{B}))\Big|_{\alpha=0}$$

$$\Rightarrow \frac{\partial\Psi}{\partial\mathbf{G}}\cdot\mathbf{B} = \frac{\partial\Upsilon}{\partial\mathbf{K}}\cdot\mathbf{F}^T\mathbf{B} = \mathbf{F}\frac{\partial\Upsilon}{\partial\mathbf{K}}\cdot\mathbf{B} \quad \forall\, \mathbf{B}\in L(\mathbb{E}^2,\mathbb{E}^3)$$

$$\Rightarrow \boxed{\boldsymbol{\mu} = \mathbf{F}\frac{\partial\Upsilon}{\partial\mathbf{K}}}$$

$$\mu_\alpha = \mathbf{F}\frac{\partial\Upsilon}{\partial\mathbf{K}}\, e_\alpha$$

$$\boxed{M_\alpha = n \times F\, \frac{\partial\Upsilon}{\partial\mathbf{K}}\, e_\alpha}\,.$$

## Orientation Preservation

Here we pause to note the "ghost" of local injectivity from 3-d nonl. elasticity — $\det \mathbf{F} > 0$. As with rods, we look at

$$F_c\big|_{X_3=0} = \nabla r + d\otimes e_3$$

w.r.t. $\{e_1, e_2, e_3\}$

$$\det F_c\big|_{X_3=0} = \det\begin{bmatrix} (r_{,1})_1 & (r_{,2})_1 & (d)_1 \\ (r_{,1})_2 & (r_{,2})_2 & (d)_2 \\ (r_{,1})_3 & (r_{,2})_3 & (d)_3 \end{bmatrix}$$

$$= \boxed{(r_{,1}\times r_{,2})\cdot d > 0}$$

local injectivity

## Strong Ellipticity

In the context of the Kirchhoff theory, we return to (SE) pp. 114 and 116 in the context of p. 212:

$$\hat{\Psi}(\underline{\underline{F}}_0, \underline{\underline{G}}) = \int_{-h/2}^{h/2} W(\underline{\underline{F}} + X_3 \underline{\underline{G}} + \underline{n} \otimes \underline{e}_3)\, dX_3$$

Consider a "rank-one excursion" in $\underline{\underline{F}} \in L(\mathbb{E}^2; \mathbb{E}^3)$:

$$\underline{\underline{F}} + \alpha\, \underline{a} \otimes \underline{b} \quad \text{where } \underset{\underline{a} \neq \underline{0}}{\underline{a} \in \mathbb{E}^3}, \ \underset{\underline{b} \neq \underline{0}}{\underline{b} \in \mathbb{E}^2}$$

Then $\dfrac{d^2}{d\alpha^2} \hat{\Psi}(\underline{\underline{F}} + \alpha\, \underline{a} \otimes \underline{b}, \underline{\underline{G}})\Big|_{\alpha=0}$

$$= \underline{a} \otimes \underline{b} \cdot \frac{\partial^2 \hat{\Psi}}{\partial \underline{\underline{F}}^2}(\underline{\underline{F}}, \underline{\underline{G}})[\underline{a} \otimes \underline{b}]$$

$$= \int_{-h/2}^{h/2} \underline{a} \otimes \underline{b} \cdot \frac{\partial^2 W}{\partial \underline{\underline{F}}^2}(\underline{\underline{F}}_c)[\underline{a} \otimes \underline{b}]\, dX_3 \overset{\text{3-d (SE)}}{>} 0$$

$$\Rightarrow \quad (*) \quad \boxed{\underline{a} \otimes \underline{b} \cdot \frac{\partial^2 \hat{\Psi}}{\partial \underline{\underline{F}}^2}(\underline{\underline{F}}, \underline{\underline{G}})[\underline{a} \otimes \underline{b}] > 0 \quad \forall\, \underline{a} \in \mathbb{E}^3, \underline{b} \in \mathbb{E}^2, \ \underline{a} \neq \underline{0},\ \underline{b} \neq \underline{0}.}$$

Observe that $\dfrac{\partial^2 \hat{\Psi}}{\partial \underline{\underline{F}}^2}(\underline{\underline{F}}, \overset{\text{fixed}}{\underline{\underline{G}}}) \in \mathcal{L}(L(\mathbb{E}^2; \mathbb{E}^3))$, i.e.,

$$\underline{e}_i \otimes \underline{e}_\alpha \cdot \frac{\partial^2 \hat{\Psi}}{\partial \underline{\underline{F}}^2}(\underline{\underline{F}}, \underline{\underline{G}})[\underline{e}_j \otimes \underline{e}_\beta] = \frac{\partial^2 \hat{\Psi}}{\partial F_{i\alpha} \partial F_{j\beta}}(\underline{\underline{F}}, \underline{\underline{G}}).$$

Next we consider a rank-one excursion in $\underline{\underline{G}} \in L(\mathbb{E}^2, \mathbb{E}^3)$:

$$\Psi(\underline{\underline{F}}, \underline{\underline{G}} + \alpha\, \underset{\sim}{a} \otimes \underset{\sim}{b}) = \int_{-h/2}^{h/2} W\left(\underline{\underline{F}} + X_3(\underline{\underline{G}} + \alpha\, \underset{\sim}{a} \otimes \underset{\sim}{b}) + \underset{\sim}{\nu} \otimes \underset{\sim}{e}_3\right) dX_3\,,$$

which is also an "$X_3\, \underset{\sim}{a} \otimes \underset{\sim}{b}$ excursion in $\underline{\underline{F}}$". In any case

$$\frac{d^2}{d\alpha^2} \Psi(\underline{\underline{F}}, \underline{\underline{G}} + \alpha\, \underset{\sim}{a} \otimes \underset{\sim}{b})\Big|_{\alpha=0}$$

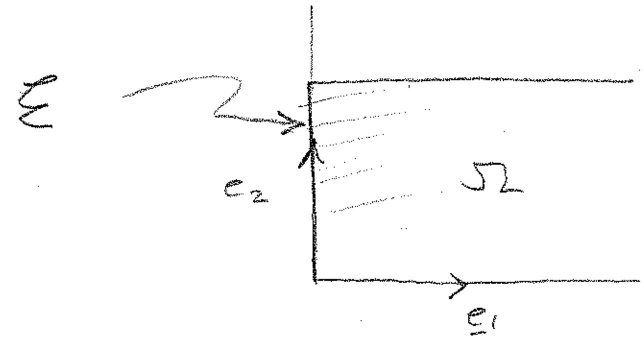
$$= \underset{\sim}{a} \otimes \underset{\sim}{b} \cdot \frac{\partial^2 \Psi}{\partial \underline{\underline{G}}^2}(\underline{\underline{F}}, \underline{\underline{G}})[\underset{\sim}{a} \otimes \underset{\sim}{b}]$$

$$= \int_{-h/2}^{h/2} X_3^2\; \underset{\sim}{a} \otimes \underset{\sim}{b} \cdot \frac{\partial^2 W}{\partial \underline{\underline{F}}^2}(\underline{\underline{F}}_c)[\underset{\sim}{a} \otimes \underset{\sim}{b}]\, dX_3 \overset{(SE)}{>} 0$$

$$\Rightarrow \quad (*) \quad \boxed{\underset{\sim}{a} \otimes \underset{\sim}{b} \cdot \frac{\partial^2 \Psi}{\partial \underline{\underline{G}}^2}(\underline{\underline{F}}, \underline{\underline{G}})[\underset{\sim}{a} \otimes \underset{\sim}{b}] > 0 \quad \forall\, \underset{\sim}{a} \in \mathbb{E}^3,\ \underset{\sim}{b} \in \mathbb{E}^2,\ \underset{\sim}{a} \neq \underset{\sim}{0},\ \underset{\sim}{b} \neq \underset{\sim}{0}.}$$

## Boundary Conditions

For simplicity, let's consider a straight edge, e.g., $\mathcal{E} = \{ X_2 \underset{\sim}{e}_2 : X_2 \in (0, \ell) \}$




$\Rightarrow r_{,2}(0, X_2) \equiv e_2$

① <u>Clamped</u>: $\quad r(0, X_2) \equiv X_2 e_2$ $\quad X_2 \in (0, \ell)$

$\boxed{r_{,1}(0, X_2) = e_1} \Rightarrow n(0, X_2) \equiv e_3$

② <u>Hinged</u>: (*) $\boxed{r(0, X_2) \equiv X_2 e_2 \quad X_2 \in (0, \ell)}$

For the remaining b.c., we resort to a variational argument to obtain the natural boundary condition:

For admissibility we require $\eta(0, X_2) \equiv 0$, $X_2 \in (0, \ell)$. From pp. 211 - 212, the term of interest to us is (since $M_\alpha = d \times \mu_\alpha$)

$$(*) \quad \delta \tilde{V} = \int_\Omega \Big[ \cdots + \overset{M_\alpha}{\frac{\partial \Phi}{\partial d_{,\alpha}}} \cdot \xi_{,\alpha} + \cdots \Big] dA = 0 \quad \forall \text{ ad } \eta, \xi$$

<u>Remark</u>: the term involving $\frac{\partial \Phi}{\partial d}$ is not of interest to us - it "falls out" of our formulation - see p. 219.

Let's examine (*) p. ~~234~~ in detail:

$$\int_\Omega \frac{\partial \Phi}{\partial \underset{\sim}{d}_{,\alpha}} \cdot \underset{\sim}{\xi}_{,\alpha} \, dA = \int_\Omega \left( \underset{\sim}{M}_1 \cdot \frac{\partial \underset{\sim}{\xi}}{\partial X_1} + \underset{\sim}{M}_2 \cdot \frac{\partial \underset{\sim}{\xi}}{\partial X_2} \right) dA$$

Along edge $\mathcal{E}$, it is the first term above that contributes to the boundary condition:

$$\int_0^\ell \int_0^{L\text{ - say}} \underset{\sim}{M}_1 \cdot \frac{\partial \underset{\sim}{\xi}}{\partial X_1} \, dX_1 \, dX_2$$

integrate by parts $\quad \underset{\sim}{U} = \underset{\sim}{M}_1 \qquad d\underset{\sim}{V} = \frac{\partial \underset{\sim}{\xi}}{\partial X_1} dX_1$

$$d\underset{\sim}{U} = \frac{\partial \underset{\sim}{M}_1}{\partial X_1} dX_1 \qquad \underset{\sim}{V} = \underset{\sim}{\xi}$$

$$= \int_0^\ell \left[ \underset{\sim}{M}_1 \cdot \underset{\sim}{\xi} \Big|_{X_1=0}^{X_1=L} - \int_0^L \frac{\partial \underset{\sim}{M}_1}{\partial X_1} \cdot \underset{\sim}{\xi} \, dX_1 \right] dX_2$$

Lecture 30

We want to examine the term at $X_1 = 0$:

$$(*) \quad \int_0^\ell \underset{\sim}{M}_1(0, X_2) \cdot \underset{\sim}{\xi}(0, X_2) \, dX_2 = 0 \quad \forall \text{ adm variations } \underset{\sim}{\xi}$$

But we need to be careful here about admissibility. Returning to (*) and (**) p. 220, we need:

$$\frac{d}{d\varepsilon} \left[ (\underset{\sim}{n} + \varepsilon \underset{\sim}{\xi}) \cdot (\underset{\sim}{n} + \varepsilon \underset{\sim}{\xi}) = 1 \right] \Bigg|_{\varepsilon = 0}$$

$$\Rightarrow \boxed{\underset{\sim}{n} \cdot \underset{\sim}{\xi} = 0}$$