Let's examine (*) p. ~~234~~ in detail:

$$\int_\Omega \frac{\partial \Phi}{\partial \underset{\sim}{d}_{,\alpha}} \cdot \underset{\sim}{\xi}_{,\alpha} \, dA = \int_\Omega \left(\underset{\sim}{\mu}_1 \cdot \frac{\partial \underset{\sim}{\xi}}{\partial X_1} + \underset{\sim}{\mu}_2 \cdot \frac{\partial \underset{\sim}{\xi}}{\partial X_2} \right) dA$$

Along edge $\mathcal{E}$, it is the first term above that contributes to the boundary condition:

$$\int_0^{\ell} \int_0^{L \text{ - say}} \underset{\sim}{\mu}_1 \cdot \frac{\partial \underset{\sim}{\xi}}{\partial X_1} \, dX_1 \, dX_2$$

(integrate by parts $\quad \underset{\sim}{u} = \underset{\sim}{\mu}_1 \qquad d\underset{\sim}{v} = \frac{\partial \underset{\sim}{\xi}}{\partial X_1} dX_1$

$$d\underset{\sim}{u} = \frac{\partial \underset{\sim}{\mu}_1}{\partial X_1} dX_1 \qquad \underset{\sim}{v} = \underset{\sim}{\xi}$$

$$= \int_0^{\ell} \left[\underset{\sim}{\mu}_1 \cdot \underset{\sim}{\xi} \Big|_{X_1 = 0}^{X_1 = L} - \int_0^L \frac{\partial \underset{\sim}{\mu}_1}{\partial X_1} \cdot \underset{\sim}{\xi} \, dX_1 \right] dX_2$$

Lecture 30

We want to examine the term at $X_1 = 0$:

$$(*) \quad \int_0^{\ell} \underset{\sim}{\mu}_1(0, X_2) \cdot \underset{\sim}{\xi}(0, X_2) \, dX_2 = 0 \quad \forall \text{ adm variations } \underset{\sim}{\xi}$$

But we need to be careful here about admissibility. Returning to (*) and (**) p. 220, we need:

$$\frac{d}{d\varepsilon} \left[(\underset{\sim}{n} + \varepsilon \underset{\sim}{\xi}) \cdot (\underset{\sim}{n} + \varepsilon \underset{\sim}{\xi}) = 1 \right] \Big|_{\varepsilon = 0}$$

$$\Rightarrow \boxed{\underset{\sim}{n} \cdot \underset{\sim}{\xi} = 0}$$

Similarly,

$$\frac{d}{d\varepsilon}\left[\left(\underset{\sim}{a}_\alpha + \varepsilon\, \underset{\sim}{\eta}_{,\alpha}\right)\cdot\left(\underset{\sim}{n} + \varepsilon\, \underset{\sim}{\xi}\right) = 0\right]\Bigg|_{\varepsilon=0}$$

$$\Rightarrow \quad \underset{\sim}{a}_\alpha \cdot \underset{\sim}{\xi} + \underset{\sim}{\eta}_{,\alpha}\cdot \underset{\sim}{n} = 0 \qquad \alpha = 1,2$$

$$\text{or} \quad \boxed{\underset{\sim}{a}_\alpha \cdot \underset{\sim}{\xi} = -\underset{\sim}{\eta}_{,\alpha}\cdot \underset{\sim}{n}} \qquad \alpha = 1,2$$

Putting these two conditions together $\Rightarrow$

(*) $\underset{\sim}{\xi} = -\left(\underset{\sim}{\eta}_{,\alpha}\cdot\underset{\sim}{n}\right)\underset{\sim}{a}^\alpha$ where $\{\underset{\sim}{a}^1, \underset{\sim}{a}^2\}$ is the reciprocal basis (so that $\{\underset{\sim}{a}^1, \underset{\sim}{a}^2, \underset{\sim}{n}\}$ is right-handed). Thus, (*) p. 235 becomes

$$\int_0^\ell \underset{\sim}{\mu}_1 \cdot \underset{\sim}{a}^\alpha \left(\underset{\sim}{\eta}_{,\alpha}\cdot\underset{\sim}{n}\right)\Bigg|_{X_1=0} dX_2$$

Finally observe that admissibility for the geometric condition ② p. 229 is

$$\underset{\sim}{\eta}(0, X_2) \equiv \underset{\sim}{0} \qquad X_2 \in (0,\ell)$$

$$\Rightarrow \underset{\sim}{\eta}_{,2}(0, X_2) \equiv \underset{\sim}{0}$$

$$\therefore \int_0^\ell \left(\underset{\sim}{\mu}_1 \cdot \underset{\sim}{a}^1\right)\left(\underset{\sim}{n}\cdot\underset{\sim}{\eta}_{,1}\right)\Bigg|_{X_1=0} dX_2 = 0 \qquad \forall \text{ adm. } \underset{\sim}{\eta}_{,1}\cdot$$

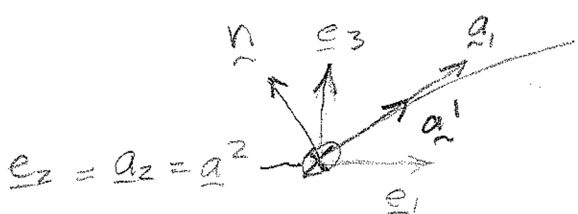
Since $\xi_1 = -\underset{\sim}{\eta}_{,1}\cdot\underset{\sim}{n}$ is arbitrary, we conclude that

$$(*) \quad \underset{\sim}{a}^1 \cdot \underset{\sim}{\mu}_1 \Big|_{X_1 = 0} = 0$$

(or we could prescribe this quantity as something non-zero.)

Observe from ② p. 234 that

$$\underset{\sim}{a}_2(0, X_2) = \underset{\sim}{r}_{,2}(0, X_2) \equiv \underset{\sim}{e}_2 = \underset{\sim}{a}^2(0, X_2)$$



Define $\hat{\underset{\sim}{a}}_1 = \dfrac{\underset{\sim}{a}_1}{|\underset{\sim}{a}_1|} = \dfrac{\underset{\sim}{a}^1}{|\underset{\sim}{a}^1|}$ at $X_1 = 0$.

Then $(*)$ above is equivalent to $\hat{\underset{\sim}{a}}_1 \cdot \underset{\sim}{\mu}_1 \Big|_{X_1=0} = 0$

Write $\underset{\sim}{\mu}_1 = (\hat{\underset{\sim}{a}}_1 \cdot \underset{\sim}{\mu}_1)\hat{\underset{\sim}{a}}_1 + (\underset{\sim}{e}_2 \cdot \underset{\sim}{\mu}_1)\underset{\sim}{e}_2 + (\underset{\sim}{n} \cdot \underset{\sim}{\mu}_1)\underset{\sim}{n} \Big|_{X_1=0}$

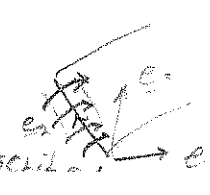
Then $\underset{\sim}{M}_1 \Big|_{X_1=0} = \underset{\sim}{n} \times \underset{\sim}{\mu}_1 \Big|_{X_1=0}$

$$= \Big[\underline{(\hat{\underset{\sim}{a}}_1 \cdot \underset{\sim}{\mu}_1)\underset{\sim}{e}_2} - (\underset{\sim}{e}_2 \cdot \underset{\sim}{\mu}_1)\hat{\underset{\sim}{a}}_1 \Big]_{X_1=0} = \Big[(\hat{\underset{\sim}{a}}_1 \cdot \underset{\sim}{M}_1)\hat{\underset{\sim}{a}}_1 + \underline{(\underset{\sim}{e}_2 \cdot \underset{\sim}{M}_1)\underset{\sim}{e}_2} \Big]_{X_1=0}$$

Hence $(*)$ is equivalent to

$$(**) \quad \underset{\sim}{e}_2 \cdot \underset{\sim}{M}_1 \Big|_{X_1=0} = 0$$

or prescribed



bending moment dist at end

③ Free Edge:

In this case, no geometric b.c.'s are prescribed. Accordingly, there are two terms of interest coming from the 1st variation:
(p. 222)

$$(*)\quad \delta \tilde{V} = \int_{\Omega} \Big(\overbrace{\frac{\partial \Phi}{\partial \underset{\sim}{r}_{,\alpha}}}^{\underset{\sim}{N}_\alpha} + q_\alpha \underset{\sim}{n} \Big) \cdot \underset{\sim}{\eta}_{,\alpha} + \overbrace{\frac{\partial \Phi}{\partial \underset{\sim}{d}_{,\alpha}}}^{\underset{\sim}{\mu}_\alpha} \cdot \underset{\sim}{\xi}_{,\alpha} + \ldots$$

Next we integrate by parts (as before p. 235):

$$\int_0^\ell \int_0^L \big(\bar{\underset{\sim}{N}}_1 + q_1 \underset{\sim}{n} \big) \cdot \frac{\partial \underset{\sim}{\eta}}{\partial X_1} + \cdots + \underset{\sim}{\mu}_1 \cdot \frac{\partial \underset{\sim}{\xi}}{\partial X_1} + \cdots \, dX_1 \, dX_2$$

$$= \int_0^\ell \Big[\big(\bar{\underset{\sim}{N}}_1 + q_1 \underset{\sim}{n} \big) \cdot \underset{\sim}{\eta} + \underset{\sim}{\mu}_1 \cdot \underset{\sim}{\xi} \Big] \Big|_{X_1 = 0}^{X_1 = L} - \cdots = 0 \quad \forall \text{ adm } \underset{\sim}{\eta}, \underset{\sim}{\xi}$$

Once again, admissibility demands that we employ (*) p. 236, which leads to

$$\int_0^\ell \big(\bar{\underset{\sim}{N}}_1 + q_1 \underset{\sim}{n} \big) \cdot \underset{\sim}{\eta} - \big(\underset{\sim}{\mu}_1 \cdot \underset{\sim}{a}^\alpha \big) \big(\underset{\sim}{\eta}_{,\alpha} \cdot \underset{\sim}{n} \big) \Big|_{X_1 = 0} dX_2$$

$$= \int_0^\ell \Big[\big(\bar{\underset{\sim}{N}}_1 + q_1 \underset{\sim}{n} \big) \cdot \underset{\sim}{\eta} - \big(\underset{\sim}{\mu}_1 \cdot \underset{\sim}{a}^1 \big) \underset{\sim}{n} \cdot \underset{\sim}{\eta}_{,1} - \big(\underset{\sim}{\mu}_1 \cdot \underset{\sim}{a}^2 \big) \underset{\sim}{n} \cdot \frac{\partial \underset{\sim}{\eta}}{\partial X_2} \Big] dX_2$$

As before $\varsigma_1 = -\underset{\sim}{n} \cdot \hat{\eta}_{,1}$ is arbitrary, and thus

$$(*) \qquad \underset{\sim}{a}^1 \cdot \underset{\sim}{\mu}_1 \Big|_{X_1 = 0} = 0 \quad \text{(or prescribed)}$$

We now integrate the third term in the integrand by parts – in this case the boundary terms are not of interest (influence other b.c.'s). We are then left with

$$\int_0^{\ell} \left[\left(\underset{\sim}{\bar{N}}_1 + q_1 \underset{\sim}{n} \right) \cdot \underset{\sim}{\eta} + \frac{\partial}{\partial X_2} \left[\left(\underset{\sim}{a}^2 \cdot \underset{\sim}{\mu}_1 \right) \underset{\sim}{n} \right] \cdot \underset{\sim}{\eta} \right] = 0$$

$\forall$ adm η

Since $\underset{\sim}{\eta}$ is arbitrary, we conclude

$$(**) \qquad \boxed{ \underset{\sim}{\bar{N}}_1 + q_1 \underset{\sim}{n} + \frac{\partial}{\partial X_2} \left[\left(\underset{\sim}{a}^2 \cdot \underset{\sim}{\mu}_1 \right) \underset{\sim}{n} \right] \Big|_{X_1 = 0} = \underset{\sim}{0} }$$

or prescribed

Note that $\frac{\partial}{\partial X_2} \left(\underset{\sim}{a}^2 \cdot \underset{\sim}{\mu}_1 \right) \underset{\sim}{n}$ combines with $q_1 \underset{\sim}{n}$ for the total "apparent" shear. This is the nonlinear version of Kirchhoff's famous result.

As before we can cast (*) and (**) in terms of the moment M_1:

Write

$$\boldsymbol{\mu}_1 = (\boldsymbol{a}^1\cdot\boldsymbol{\mu}_1)\boldsymbol{a}_1 + (\boldsymbol{a}^2\cdot\boldsymbol{\mu}_1)\boldsymbol{a}_2 + (\boldsymbol{n}\cdot\boldsymbol{\mu}_1)\boldsymbol{n}$$

Then $\boldsymbol{M}_1 = \boldsymbol{n}\times\boldsymbol{\mu}_1$

$$= (\boldsymbol{a}^1\cdot\boldsymbol{\mu}_1)\,\boldsymbol{n}\times\boldsymbol{a}_1 + (\boldsymbol{a}^2\cdot\boldsymbol{\mu}_1)\,\boldsymbol{n}\times\boldsymbol{a}_2$$

It's not hard to see that

$$\boldsymbol{n}\times\boldsymbol{a}_1 = J\,\boldsymbol{a}^2 \qquad J \equiv |\boldsymbol{a}_1\times\boldsymbol{a}_2| \ (p.\,220)$$

and

$$\boldsymbol{n}\times\boldsymbol{a}_2 = -J\,\boldsymbol{a}^1$$

(Indeed, $\boldsymbol{a}^1\cdot\boldsymbol{a}_2 = \boldsymbol{a}^2\cdot\boldsymbol{a}_1 = 0$, and

$$\boldsymbol{a}^1\cdot\boldsymbol{a}_1 = \frac{1}{J}(\boldsymbol{a}_2\times\boldsymbol{n})\cdot\boldsymbol{a}_1 = \frac{1}{J}(\boldsymbol{a}_1\times\boldsymbol{a}_2)\cdot\boldsymbol{n} = 1 \checkmark$$

$$\boldsymbol{a}^2\cdot\boldsymbol{a}_2 = \frac{1}{J}(\boldsymbol{n}\times\boldsymbol{a}_1)\cdot\boldsymbol{a}_2 = \frac{1}{J}(\boldsymbol{a}_1\times\boldsymbol{a}_2)\cdot\boldsymbol{n} = 1 \checkmark \Big)$$

$$\therefore\ \boldsymbol{M}_1 = (\boldsymbol{a}^1\cdot\boldsymbol{\mu}_1)J\,\boldsymbol{a}^2 - (\boldsymbol{a}^2\cdot\boldsymbol{\mu}_1)J\,\boldsymbol{a}^1$$

$$\Rightarrow \begin{cases} \boldsymbol{a}_1\cdot\boldsymbol{M}_1 = -J\,\boldsymbol{a}^2\cdot\boldsymbol{\mu}_1 \\ \boldsymbol{a}_2\cdot\boldsymbol{M}_1 = J\,\boldsymbol{a}^1\cdot\boldsymbol{\mu}_1 \end{cases}$$

So $(*) \Leftrightarrow$

$$\boldsymbol{a}_2\cdot\boldsymbol{M}_1\Big|_{X_1=0} = 0$$

(or prescribed

prescribed bending moment at edge.)

while $(**)$ is equivalent to

$$\left\{ \bar{\underset{\sim}{N}}_1 + q_1 \underset{\sim}{n} - \frac{\partial}{\partial X_2}\left[\left(\frac{\underset{\sim}{a}_1 \cdot \underset{\sim}{M}_1}{J} \right) \underset{\sim}{n} \right] \right\} \Bigg|_{X_1 = 0} = 0$$

or prescribed

<u>Note:</u> $\left. q_1 - \frac{\partial}{\partial X_2}\left(\frac{\underset{\sim}{a}_1 \cdot \underset{\sim}{M}_1}{J} \right) \right|_{X_1=0}$ is the apparent net shear dist.

while $\left. \left(\bar{\underset{\sim}{N}}_1 - \left(\frac{\underset{\sim}{a}_1 \cdot \underset{\sim}{M}_1}{J} \right) \frac{\partial \underset{\sim}{n}}{\partial X_2} \right) \right|_{X_1=0}$

is the apparent in-plane force distr.