subject to the pointwise constraint

$$\det \underset{\sim}{\nabla} \underset{\sim}{f} \equiv 1$$

Lagrange-multiplier method:

Find stationary values of

$$\tilde{V}(\underset{\sim}{f}, p) = V(\underset{\sim}{f}) - \int_{\mathcal{B}} \overset{\text{Lagrange mult.}}{p} [\det(\nabla \underset{\sim}{f}) - 1] \, dV$$

Motivation: Suppose $V(\underset{\sim}{q})$, $V : \mathbb{R}^n \to \mathbb{R}$, is to be minimized, subject to the constraint $\underset{\sim}{g}(\underset{\sim}{q}) = \underset{\sim}{0}$, where $\underset{\sim}{g} : \mathbb{R}^n \to \mathbb{R}^m$, $m < n$. Then the Lagrange mult. method reads, find stationary values of

$$\tilde{V}(\underset{\sim}{q}, \underset{\sim}{\lambda}) \equiv V(\underset{\sim}{q}) - \underbrace{\langle \overset{\text{vector-valued Lagr. mult.}}{\underset{\sim}{\lambda}}, \underset{\sim}{g}(\underset{\sim}{q}) \rangle}_{\text{inner product on } \mathbb{R}^m}$$

In our problem $\det(\underset{\sim}{\nabla} \underset{\sim}{f}) - 1$ plays the role of "$\underset{\sim}{g}$" and p plays the role of $\underset{\sim}{\lambda}$.

Moreover, $\int_{\mathcal{B}} y(\underset{\sim}{X}) \, z(\underset{\sim}{X}) \, dV$ defines an inner product on functions $y : \bar{\mathcal{E}} \to \mathbb{R}$.

In any case, let's take the first variation of $\tilde{V}(\underset{\sim}{f}, p)$:

$$\delta \tilde{V} = \frac{d}{d\alpha} \tilde{V}\left(\underset{\sim}{f} + \alpha \underset{\sim}{\eta},\ p + \alpha \xi\right)\Big|_{\alpha=0}$$

$$= \frac{d}{d\alpha}\Bigg\{ \int_{\mathcal{B}} W\left(\nabla \underset{\sim}{f} + \alpha \nabla \underset{\sim}{\eta}\right) dV$$

$$- \left[\int_{\mathcal{B}} \hat{\underset{\sim}{b}} \cdot \left(\underset{\sim}{f} + \alpha \underset{\sim}{\eta}\right) dV + \int \hat{\underset{\sim}{s}} \cdot \left(\underset{\sim}{f} + \alpha \underset{\sim}{\eta}\right) ds \right]$$

$$- \int_{\mathcal{B}} (p + \alpha \xi) \left[\det\left(\nabla \underset{\sim}{f} + \alpha \nabla \underset{\sim}{\eta}\right) - 1\right] dV \Bigg\}\Bigg|_{\alpha=0}$$

$$= \Bigg(\int_{\mathcal{B}} \frac{dW}{d\underset{\approx}{F}}\left(\nabla \underset{\sim}{f}\right) \cdot \nabla \underset{\sim}{\eta}\, dV$$

$$- \left[\int_{\mathcal{B}} \hat{\underset{\sim}{b}} \cdot \underset{\sim}{\eta}\, dV + \int_{\partial \mathcal{B}_2} \hat{\underset{\sim}{s}} \cdot \underset{\sim}{\eta}\, ds \right]$$

weak form of governing equations: equil. + constraint

$$- \int_{\mathcal{B}} \xi \left[\det\left(\nabla \underset{\sim}{f}\right) - 1\right] dV$$

$$- \int_{\mathcal{B}} p \ \mathrm{Cof}\left(\nabla \underset{\sim}{f}\right) \cdot \nabla \underset{\sim}{\eta}\, dV = 0$$

$\forall$ adm $\underset{\sim}{\eta}$, ξ

∴ 1st choose $\underset{\sim}{\eta} \equiv \underset{\sim}{0} \Rightarrow$

weak ~~form~~ form of constraint: $\boxed{\int_{\mathcal{B}} [\det(\nabla \underline{f}) - 1]\, \xi \, dV = 0} \quad \forall\, \xi$ contin.

$$\Rightarrow \det(\nabla \underline{f}) \equiv 1 \quad (\underline{f} \text{ suff smooth})$$

Then $$\delta \tilde{V} = \int_{\mathcal{B}} \left[\frac{dW}{d\underline{\underline{F}}}(\nabla \underline{f}) - p \operatorname{Cof}(\nabla \underline{f})\right] \cdot \nabla \underline{\eta}\, dV$$

$$- \left[\int_{\mathcal{B}} \hat{\underline{b}} \cdot \underline{\eta}\, dV + \int_{\partial \mathcal{B}_2} \hat{\underline{s}} \cdot \underline{\eta}\, ds\right] = 0$$

weak form of equilibrium $\forall$ adm $\underset{\sim}{\eta}$

Assuming smooth $\underline{f}$, lets proceed to the strong form:

Call $$\frac{dW}{d\underline{\underline{F}}}(\nabla \underline{f}) - p \operatorname{Cof}(\nabla \underline{f}) \equiv \underline{\underline{S}}$$

Then (p. 81) $$\underline{\underline{S}} \cdot \nabla \underline{\eta} = \underline{\nabla} \cdot (\underline{\underline{S}}^T \underline{\eta}) - (\underline{\nabla} \cdot \underline{\underline{S}}) \cdot \underline{\eta}$$

$$\therefore \int_{\mathcal{B}} \underline{\underline{S}} \cdot \nabla \underline{\eta} = \int_{\mathcal{B}} \left[\underset{\sim}{\nabla} \cdot (\underline{\underline{S}}^T \underline{\eta}) - (\underline{\nabla} \cdot \underline{\underline{S}}) \cdot \underline{\eta}\right] dV$$

$$\overset{\text{div thm}}{=} \int_{\substack{\partial \mathcal{B} \\ \partial \mathcal{B}_2}} \underline{\underline{S}}^T \underset{\sim}{\eta} \cdot \underset{\sim}{n}\, ds - \int_{\mathcal{B}} (\underset{\sim}{\nabla} \cdot \underline{\underline{S}}) \cdot \underline{\eta}\, dV$$

$$(\underset{\sim}{\eta}|_{\partial \mathcal{B}_1} = \underset{\sim}{0})$$

$$\therefore \quad \delta \tilde{V} = -\int_{\mathcal{B}} \left(\nabla \cdot \underline{\underline{S}} + \hat{\underline{b}} \right) \cdot \underset{\sim}{\eta} \, dV$$

$$+ \int_{\partial \mathcal{B}_2} \left(\underline{\underline{S}} \, \underset{\sim}{m} - \hat{\underset{\sim}{s}} \right) \cdot \underset{\sim}{\eta} \, ds = 0$$

$$\forall \text{ adm } \underset{\sim}{\eta}$$

Choose $\underset{\sim}{\eta}\big|_{\partial \mathcal{B}_2} = \underset{\sim}{0}$ (perfectly admissible!)

(*) Then $\boxed{\nabla \cdot \left[\frac{\partial W}{\partial \underline{\underline{F}}}(\nabla \underline{f}) - p \, \mathrm{Cof}(\nabla \underline{f}) \right] + \hat{\underset{\sim}{b}} = \underset{\sim}{0}}$

Next suppose $\underset{\sim}{\eta}\big|_{\partial \mathcal{B}_2} \neq \underset{\sim}{0} \Rightarrow$

$$\boxed{\left[\frac{\partial W}{\partial \underline{\underline{F}}}(\nabla \underline{f}) - p \, \mathrm{Cof}(\nabla \underline{f}) \right] \underset{\sim}{m} = \hat{\underset{\sim}{s}} \quad \text{on } \partial \mathcal{B}_2}$$

Remark. If we expand the 2^{nd} term in (*) above (taking the divergence), we get

$$- p \, \nabla \cdot \left(\mathrm{Cof}(\nabla \underline{f}) \right) - [\mathrm{Cof} \, \nabla \underline{f}] \nabla p .$$

However, we claim that

$$\nabla \cdot \left(\mathrm{Cof} \, \nabla \underline{f} \right) = \underset{\sim}{0}$$

To see this, first recall the scalarversion of the divergence theorem:

$$\int_{\partial \mathcal{P}} \phi \, \underset{\sim}{n} \, ds = \int_{\mathcal{P}} \underline{\nabla} \phi \, dV$$

for all smooth fields $\phi(\underline{X})$ and domains $\mathcal{P}$ with sufficiently smooth boundaries $\partial\mathcal{P}$.

Next consider

$$\underbrace{\int_{\partial\mathcal{P}} \underset{\sim}{n} \, ds}_{0} = \int_{\partial \mathcal{P}_0} \mathrm{Cof}(\underline{\nabla} \underline{f}) \, \underset{\sim}{m} \, dS$$

$$\overset{\text{div thm}}{=} \int_{\mathcal{P}_0} \nabla \cdot \left(\mathrm{Cof}(\underline{\nabla} \underline{f})\right) dV \quad \forall \mathcal{P}_0 \subset \mathcal{B}_0$$

But by the scalar div thm ($\phi \equiv 1$) the left side above is zero. Since this holds for all $\mathcal{P}_0 \subset \mathcal{B}_0$, we deduce the claim.

Accordingly, the local linear-momentum balance equation reduces to

$$\nabla \cdot \left(\frac{\partial W}{\partial \underline{F}}(\nabla f)\right) - \mathrm{Cof}(\underline{\nabla} \underline{f}) \, \underset{\sim}{\nabla} p + \underset{\sim}{b} = 0$$

subject to $\det(\underline{\nabla} \underline{f}) \equiv 1$.

Constitutive Hypotheses

Recall the classical field equations (for the dead-load problem) for elastostatics:

$$
* \begin{cases}
\nabla \cdot \underline{\underline{S}} + \hat{\underset{\sim}{b}} = \underset{\sim}{0} & \text{in } \mathcal{B}, \\
\underset{\sim}{f} = \hat{\underset{\sim}{f}} & \text{on } \partial\mathcal{B}_1, \\
\underline{\underline{S}}\,\underset{\sim}{m} = \hat{\underset{\sim}{s}} & \text{on } \partial\mathcal{B}_2, \\
\text{where } \quad \underline{\underline{S}} = \dfrac{\partial W}{\partial \underline{\underline{F}}}(\underline{\underline{F}}), & \text{and where the stored-}
\end{cases}
$$

energy function W is subject to the requirements of frame indifference & material symmetry (usually isotropy).

Central Question: (Truesdell's "Haupt Problem")
What additional, physically reasonable restrictions should be placed upon W to insure existence of solutions for a broad class of problems?

Motivation from the calculus of variations

Seek minimizers (local or global = stable equilibria) of the potential energy

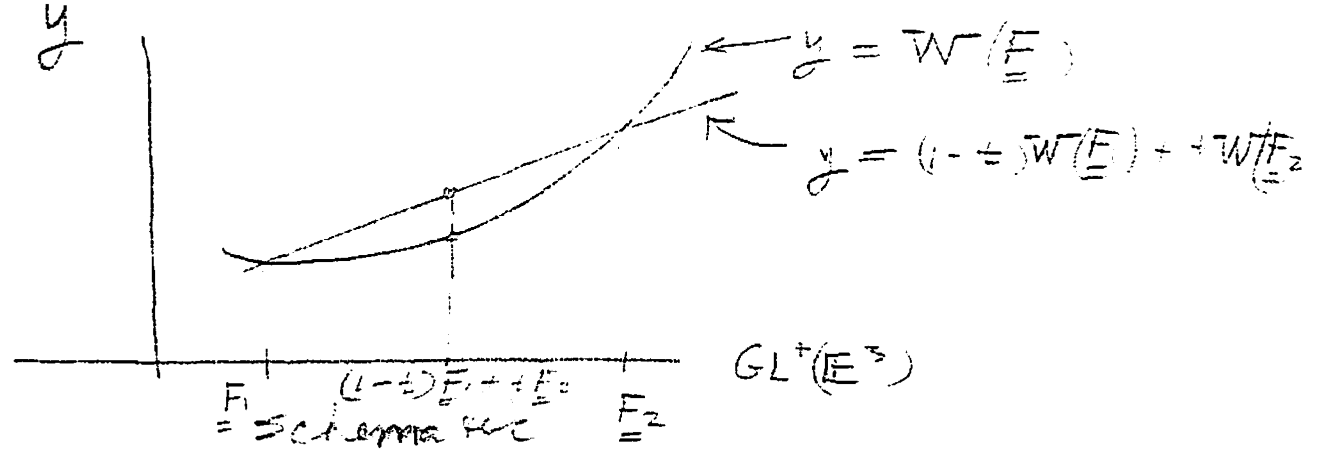
$$
V[\underset{\sim}{f}] = \int_{\mathcal{B}} W(\nabla \underset{\sim}{f})\,dV - \left[\int_{\mathcal{B}} \hat{\underset{\sim}{b}} \cdot \underset{\sim}{f}\,dV + \int_{\partial\mathcal{B}_2} \hat{\underset{\sim}{s}} \cdot \underset{\sim}{f}\,dV\right]
$$

From a purely mathematical point of view, this minimization problem is in *very good* *shape** if $W(\cdot)$ is convex, i.e.,

$$W\big((1-t)\underline{\underline{F}}_1 + t\underline{\underline{F}}_2\big) \underset{(<)}{\leq} (1-t)W(\underline{\underline{F}}_1) + tW(\underline{\underline{F}}_2)$$

$$\forall\ \underline{\underline{F}}_1, \underline{\underline{F}}_2 \in GL^+(\mathbb{E}^3),\ t \in [0,1],$$

such that $(1-t)\underline{\underline{F}}_1 + t\underline{\underline{F}}_2 \in GL^+(\mathbb{E}^3)$



*In other words, there is an $\underline{f}^*$ such that
$V[\underline{f}^*] \leq V[\underline{f}]$ for all $\underline{f}$ in some class of functions ($W^{1,p}$)

But there are many reasons why $W(\cdot)$ convex is not *physically realistic*:

① If $W(\cdot)$ is *strictly* convex ($<$), then there is at most *one* *solution* to (+), which, as we have already discussed, is not realistic. (We omit the proof, due to R. Hill, which is not hard (but not very illuminating).)

(2) Convexity of $\underline{\underline{F}} \mapsto W(\cdot)$ violates <u>material objectivity</u>: To see this, go ~~back~~ to the definition of convexity on p. 92:

$$W\big((1-t)\underline{\underline{F}}_1 + t\underline{\underline{F}}_2\big) - W(\underline{\underline{F}}_1) \leq t\left[W(\underline{\underline{F}}_2) - W(\underline{\underline{F}}_1)\right]$$

$$\Rightarrow \lim_{t \searrow 0} \underbrace{\frac{W\big((1-t)\underline{\underline{F}}_1 + t\underline{\underline{F}}_2\big) - W(\underline{\underline{F}}_1)}{t}}_{\frac{\partial W}{\partial \underline{\underline{F}}_1}(\underline{\underline{F}}_1)\cdot(\underline{\underline{F}}_2 - \underline{\underline{F}}_1)} \leq W(\underline{\underline{F}}_2) - W(\underline{\underline{F}}_1)$$

$$\frac{\partial W}{\partial \underline{\underline{F}}_1}(\underline{\underline{F}}_1)\cdot(\underline{\underline{F}}_2 - \underline{\underline{F}}_1) \leq W(\underline{\underline{F}}_2) - W(\underline{\underline{F}}_1)$$

$$\forall\ \underline{\underline{F}}_1, \underline{\underline{F}}_2 \in GL^+(\mathbb{E}^3)$$

Now let $\underline{\underline{F}}_1 = \underline{\underline{F}}$ and $\underline{\underline{F}}_2 = \underline{\underline{Q}}\,\underline{\underline{F}}$ for $\underline{\underline{Q}} \in SO(3)$.

Then $\displaystyle \frac{dW}{d\underline{\underline{F}}}(\underline{\underline{F}})\cdot(\underline{\underline{Q}}\,\underline{\underline{F}} - \underline{\underline{F}}) \leq \underbrace{W(\underline{\underline{Q}}\,\underline{\underline{F}}) - W(\underline{\underline{F}})}_{W(\underline{\underline{F}}) - W(\underline{\underline{F}}) = 0}$

$\therefore \quad \hat{\underline{\underline{S}}}(\underline{\underline{F}})\,\underline{\underline{F}}^T\cdot(\underline{\underline{Q}} - \underline{\underline{I}}) \leq 0 \qquad \forall\ \underline{\underline{Q}} \in SO(3)$

But $\quad \hat{\underline{\underline{S}}}(\underline{\underline{F}}) = \det\underline{\underline{F}}\ \hat{\underline{\underline{T}}}(\underline{\underline{F}})\,\underline{\underline{F}}^{-T}$

$\therefore \quad \hat{\underline{\underline{T}}}(\underline{\underline{F}})\cdot(\underline{\underline{Q}} - \underline{\underline{I}}) \leq 0 \qquad \forall\ \underline{\underline{Q}} \in SO(3)$

In particular, choose $\underline{\underline{Q}} = -(\underline{e}_1\otimes\underline{e}_1 + \underline{e}_2\otimes\underline{e}_2) + \underline{e}_3\otimes\underline{e}_3$

Then $\quad -2\left(\hat{T}(\underline{\underline{F}})_{11} + \hat{T}(\underline{\underline{F}})_{22}\right) \leq 0$

$$\Rightarrow \quad \hat{T}(\underline{\underline{F}})_{11} + \hat{T}(\underline{\underline{F}})_{22} \geq 0$$

for all $\underline{\underline{F}} \in SL^+(\mathbb{E}^3)$, which can not be true in general. E.g., a hydrostatic state of stress, $\hat{\underline{\underline{T}}}(\underline{\underline{F}}) = \underline{\underline{T}} = -\Pi_0 \underline{\underline{I}}$, would not be possible.

Clearly $\underline{\underline{F}} \mapsto W(\underline{\underline{F}})$ convex is out. Recall $W(\underline{\underline{F}}) = \sigma(\underline{\underline{U}})$ from (MO).

③ What about $\underline{\underline{U}} \mapsto \sigma(\underline{\underline{U}})$ convex?

<u>Claim</u>: this can not be a general assumption. In particular this is bad for rubber, which is isotropic and nearly incompressible, i.e., it takes a lot of energy to deviate from $\lambda_1 \lambda_2 \lambda_3 = 1$. For simplicity, take $\lambda_3 = 1$, and consider bi-axial stretching viz., $\underline{\underline{F}} = \underline{\underline{U}} = \sum_{\alpha=1}^{2} \lambda_\alpha \underline{e}_\alpha \otimes \underline{e}_\alpha + \underline{e}_3 \otimes \underline{e}_3$.

Before proceeding on with our argument, we need the following result for isotropic materials:

Write: $\quad W(\underline{\underline{U}}) = \Phi(I_B, II_B, III_B)$

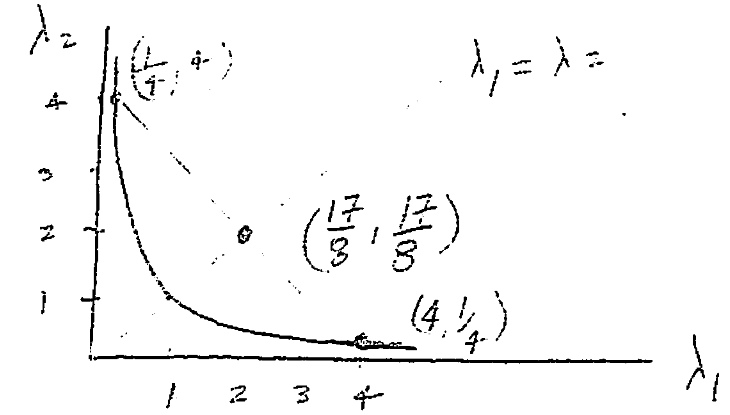
From the calculations on p. 73, we deduce:

$$I_B = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$
$$II_B = (\lambda_1\lambda_2)^2 + (\lambda_2\lambda_3)^2 + (\lambda_1\lambda_3)^2$$
$$III_B = (\lambda_1\lambda_2\lambda_3)^2$$

$$\therefore \quad \sigma(\underline{\underline{V}}) = \tilde{\sigma}(\lambda_1, \lambda_2, \lambda_3).$$

Observe that $\tilde{\sigma}(\lambda_{p_1}, \lambda_{p_2}, \lambda_{p_3}) \equiv \sigma(\lambda_1, \lambda_2, \lambda_3)$ for all permutations (p_1, p_2, p_3) of $(1, 2, 3)$.

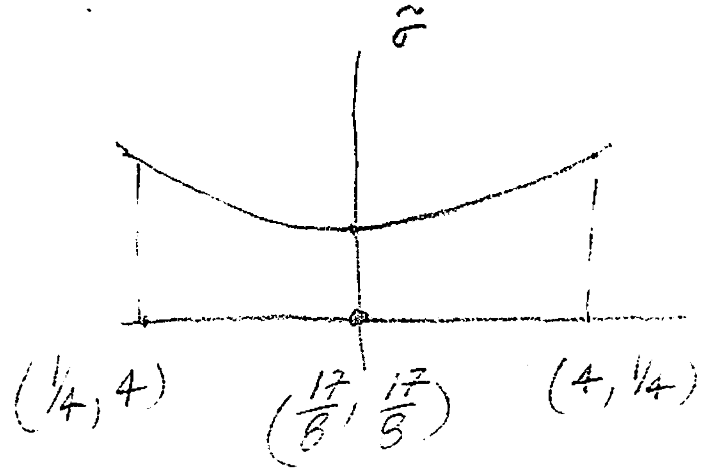
Back to our "arm-chair" experiment: For $\lambda_3 = 1$, it is "relatively easy" to deform the body along $\lambda_1\lambda_2 = 1$ (low energy) and "hard" to move away from it (high energy).



Thus, $\tilde{\sigma}(\frac{17}{8}, \frac{17}{8}, 1) > \tilde{\sigma}(4, \frac{1}{4}, 1) \overset{\text{isotropy}}{=} \tilde{\sigma}(\frac{1}{4}, 4, 1)$

But this is <u>incompatible</u> with convexity:

Indeed, if $\underline{v} \mapsto \sigma(\underline{v})$ is convex, then so too is $(\lambda_1, \lambda_2, \lambda_3) \longmapsto \tilde{\sigma}(\lambda_1, \lambda_2, \lambda_3)$, and along the line segment connecting $(1/4, 4)$ to $(4, 1/2)$ (in the figure on p. 95) $\tilde{\sigma}(\cdot)$ then would look like:



Thus, $\tilde{\sigma}\left(\frac{17}{8}, \frac{17}{8}, 1\right) \leq \tilde{\sigma}(4, 1/4, 1) = \tilde{\sigma}(1/4, 4, 1)$.

Instead of the calculus of variations, one can turn to pde ~~theory~~ for motivation: The principal part of differential operator (from p. 91) comes from

$$(*) \qquad \underset{\sim}{\nabla} \cdot \left(\frac{dW}{d\underline{\underline{F}}} (\underline{\nabla f}) \right),$$

which we now expand out: Wrt an orthonormal basis we have

$$S_{ij} = \frac{\partial W}{\partial F_{ij}} (F_{k\ell})$$

$F_{k\ell}$

$$\text{Then } (\underline{\nabla} \cdot \underline{\underline{S}})_i = \frac{\partial}{\partial X_j} \left(\frac{\partial W}{\partial F_{ij}} \left(\frac{\partial f_k}{\partial X_\ell} \right) \right)$$

$$\therefore \quad (\underline{\nabla} \cdot \underline{\underline{S}})_i = \frac{\partial^2 W}{\partial F_{ij}\,\partial F_{kl}}\left(\frac{\partial f_k}{\partial X_l}\right)\frac{\partial^2 f_k}{\partial X_j\,\partial X_l} \quad (**)$$

<u>Defn</u> The 4^{th} order tensor

$$\underset{\sim}{\mathbb{C}}(\underline{\underline{F}}) = \frac{\partial^2 W}{\partial \underline{\underline{F}}^2}(\underline{\underline{F}}) \, ,$$

where $\underset{\sim}{\mathbb{C}}(\underline{\underline{F}})[\underline{\underline{A}}] \equiv \frac{d}{d\alpha}\left(\frac{\partial W}{\partial \underline{\underline{F}}}(\underline{\underline{F}} + \alpha \underline{\underline{A}})\right)\Big|_{\alpha=0}$

$\forall \; \underline{\underline{A}} \in L(E^3)$

is called the <u>elasticity</u> <u>tensor</u> at $\underline{\underline{F}}$.

<u>Claim</u> $\mathbb{C}_{ijkl}(\underline{\underline{F}}) \equiv \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \cdot \underset{\sim}{\mathbb{C}}(\underline{\underline{F}})[\underset{\sim}{e}_k \otimes \underset{\sim}{e}_l]$

$$= \frac{\partial^2 W}{\partial F_{ij}\,\partial F_{kl}}$$

Now as defined, $\underset{\sim}{\mathbb{C}}(\underline{\underline{F}}) : L(E^3) \to L(E^3)$
i.e., $\underset{\sim}{\mathbb{C}}(\underline{\underline{F}}) \in L(L(E^3))$ (for each $\underline{\underline{F}}$)

Thus, with basis $\{ \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j : i,j = 1,2,3 \}$
for $L(E^3)$, we have

$$\mathbb{C}_{ijkl}(\underline{\underline{F}}) = \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \cdot \underset{\sim}{\mathbb{C}}(\underline{\underline{F}})[\underset{\sim}{e}_k \otimes \underset{\sim}{e}_l]$$