Lecture 12

and $\underset{\sim}{\mathbb{C}}(\underset{=}{F}) = \mathbb{C}_{ijkl}(\underset{=}{F})\,(\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j) \otimes (\underset{\sim}{e}_k \otimes \underset{\sim}{e}_l)$

(parentheses not needed)

In this way we can establish

$$\underset{\sim}{\mathbb{C}}(\underset{=}{F}) \in L(L(\mathbb{E}^3)) \qquad \text{(we haven't named!)}$$

$$\simeq L(\text{3rd-order tensors}, \mathbb{E}^3)$$

$$\simeq L(\mathbb{E}^3, \text{3rd-order tensors}), \text{ e.g.}$$

E.g.,

$$\underset{\sim}{\mathbb{C}}\,\underset{\sim}{v} = \mathbb{C}_{ijkl}\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \otimes \underset{\sim}{e}_k \otimes \underset{\sim}{e}_l \; v_m \underset{\sim}{e}_m$$

$$= \mathbb{C}_{ijkl}\, v_m\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \otimes \underset{\sim}{e}_k\, \delta_{lm}$$

$$= \mathbb{C}_{ijkl}\, v_l\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \otimes \underset{\sim}{e}_k$$

(3rd order tensor for fixed $\underset{\sim}{v}$)

Next, consider

$$\frac{d}{d\alpha} \frac{\partial W}{\partial F_{ij}}\left(\underset{=}{F} + \alpha\, \underset{\sim}{e}_1 \otimes \underset{\sim}{e}_3\right)\Big|_{\alpha=0}$$

$$= \frac{d}{d\alpha}\left(\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \cdot \frac{\partial W}{\partial \underset{=}{F}}\left(\underset{=}{F} + \alpha\, \underset{\sim}{e}_1 \otimes \underset{\sim}{e}_3\right)\right)\Big|_{\alpha=0}$$

$\left(\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \cdot \underset{=}{A} = \underset{\sim}{e}_i \cdot \underset{=}{A}\underset{\sim}{e}_j\right)$

$$= \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \cdot \frac{\partial^2 W}{\partial F^2}(\underset{=}{F})\left[\underset{\sim}{e}_1 \otimes \underset{\sim}{e}_3\right]$$

But $\dfrac{d}{d\alpha}\dfrac{\partial W}{\partial F_{ij}}\left(F_{11}, F_{12}, F_{13}+\alpha, F_{21}, \dots F_{33}\right)$

$$= \frac{\partial^2 W}{\partial F_{ij}\,\partial F_{13}}$$

$$\therefore \quad \frac{\partial^2 W}{\partial F_{ij}\partial F_{13}}(\underset{=}{F}) = \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \cdot \frac{\partial^2 W}{\partial \underset{=}{F}^2}(\underset{=}{F})[\underset{\sim}{e}_1 \otimes \underset{\sim}{e}_3]$$

$$= \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \cdot \underset{\sim}{\mathcal{C}}(\underset{=}{F})[\underset{\sim}{e}_1 \otimes \underset{\sim}{e}_3]$$

Generally, $\dfrac{\partial^2 W}{\partial F_{ij}\,\partial F_{kl}}(\underset{=}{F}) = \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \cdot \underset{\sim}{\mathcal{C}}(\underset{=}{F})[\underset{\sim}{e}_k \otimes \underset{\sim}{e}_l]$

$$= C_{ijkl}(\underset{=}{F}) \quad \checkmark$$

<u>Lemma</u> For smooth W, $\underset{\sim}{\mathcal{C}}(\underset{=}{F})$ is symmetric (for each $\underset{=}{F} \in GL^+(\mathbb{E}^3)$),

i.e.,

$$\underbrace{\underset{=}{A} \cdot \underset{\sim}{\mathcal{C}}(\underset{=}{F})[\underset{=}{B}]}_{A_{ij}\frac{\partial^2 W}{\partial F_{ij}\partial F_{kl}}B_{kl}} = \underbrace{\underset{=}{B} \cdot \underset{\sim}{\mathcal{C}}(\underset{=}{F})[\underset{=}{A}]}_{B_{ij}\frac{\partial^2 W}{\partial F_{ij}\partial F_{kl}}A_{kl}} \quad \forall\, \underset{=}{A}, \underset{=}{B} \in L(\mathbb{E}^3)$$

$$A_{ij}\frac{\partial^2 W}{\partial F_{kl}\,\partial F_{ij}}B_{kl} \quad \checkmark$$

With our new notation in hand, (**) p.97 reads:

$$\underset{\sim}{\nabla} \cdot \left(\frac{dW}{d\underset{=}{F}}(\underset{\sim}{\nabla} \underset{\sim}{f})\right) \equiv \underset{\sim}{\mathcal{C}}(\underset{\sim}{\nabla} \underset{\sim}{f})[\underset{\sim}{\nabla}^2 \underset{\sim}{f}], \quad \text{where}$$

$$\left(\underset{\sim}{\mathbb{C}}(\nabla \underline{f})[\nabla^2 \underline{f}]\right)_i \equiv \mathbb{C}_{ijkl}(\nabla \underline{f}) \frac{\partial^2 f_k}{\partial X_l \partial X_j}$$

wrto to any orthonormal basis.

We are interested in properties of $\underset{\sim}{\mathbb{C}}(\underline{\underline{F}})$ – those of $\underset{\sim}{\mathbb{C}}(\underline{\underline{I}})$ are well known. Indeed, suppose that the reference configuration is "~~stress free~~" or "natural", viz.,

$$(*)\quad \underset{\approx}{S} = \frac{\partial W}{\partial \underline{\underline{F}}}(\underline{\underline{I}}) = \underline{\underline{0}} .$$

Then (expanding) in general, we have

$$\underset{\approx}{S} = \frac{\partial W}{\partial \underline{\underline{F}}}(\underline{\underline{F}}) = \cancel{\frac{\partial W}{\partial \underline{\underline{F}}}(\underline{\underline{I}})}^{\,\underline{\underline{0}}} + \frac{\partial^2 W}{\partial \underline{\underline{F}}^2}(\underline{\underline{I}})(\underline{\underline{F}} - \underline{\underline{I}}) + \text{h.o.t.}$$

$$\text{or} \quad \underline{\underline{S}} = \underset{\sim}{\mathbb{C}}(\underline{\underline{I}})[\underline{\underline{F}} - \underline{\underline{I}}] + \text{h.o.t.},$$

$$\text{If} \quad \underline{\underline{F}} = \nabla \underline{f}, \quad \underline{\underline{F}} - \underline{\underline{I}} = \underset{\sim}{\nabla} \underset{\sim}{u} \quad \text{displacement gradient}$$

$$\text{Then} \quad \underline{\underline{S}} = \underset{\sim}{\mathbb{C}}(\underline{\underline{I}})[\underset{\sim}{\nabla} \underset{\sim}{u}] + \text{h.o.t.}$$

Thus, this is ~~the~~ elasticity tensor from the linear ~~theory~~, for which much is known:

<u>Thm</u> If the reference configuration is stress-free[(*)], then

$$\underset{\sim}{\mathbb{C}}(\underline{\underline{I}})[\underline{\underline{H}}] \in \mathrm{Sym}(\mathbb{E}^3) \quad \forall\ \underline{\underline{H}} \in L(\mathbb{E}^3),$$

$$\underset{\sim}{\mathbb{C}}(\underline{\underline{I}})[\underline{\underline{W}}] = \underline{\underline{0}} \quad \forall\ \underline{\underline{W}} \in \mathrm{Skew}(\mathbb{E}^3).$$

To prove this we first need:

<u>Lemma</u> $\quad \underset{\sim}{\mathbb{C}}(\underline{\underline{I}}) = \dfrac{d\hat{\underline{\underline{T}}}}{d\underline{\underline{F}}}(\underline{\underline{I}})$

<u>Pf</u> Write $\quad \dfrac{dW}{d\underline{\underline{F}}}(\underline{\underline{F}})$

$$\hat{\underline{\underline{S}}}(\underline{\underline{F}})\,\underline{\underline{F}}^T = \det \underline{\underline{F}}\;\hat{\underline{\underline{T}}}(\underline{\underline{F}})$$

Set $\underline{\underline{F}} = \underline{\underline{I}} + \alpha \underline{\underline{H}}$ and compute $\dfrac{d}{d\alpha}(\quad)\Big|_{\alpha=0}$

$$\frac{d}{d\alpha}\left\{\frac{dW}{d\underline{\underline{F}}}(\underline{\underline{I}}+\alpha\underline{\underline{H}})(\underline{\underline{I}}+\alpha\underline{\underline{H}}^T) = \det(\underline{\underline{I}}+\alpha\underline{\underline{H}})\,\hat{\underline{\underline{T}}}(\underline{\underline{I}}+\alpha\underline{\underline{H}})\right\}\Big|_{\alpha=0}$$

$$\Rightarrow \underset{\sim}{\mathbb{C}}(\underline{\underline{I}})[\underline{\underline{H}}]\,\underline{\underline{I}} + \cancel{\hat{\underline{\underline{S}}}(\underline{\underline{I}})}^{\underline{\underline{0}}}\,\underline{\underline{H}}^T$$

$$= \operatorname{Cof}(\underline{\underline{I}})\,\underline{\underline{H}}\,\cancel{\hat{\underline{\underline{T}}}(\underline{\underline{I}})}^{\underline{\underline{0}}} + \det(\underline{\underline{I}})\frac{d\hat{\underline{\underline{T}}}}{d\underline{\underline{F}}}(\underline{\underline{I}})[\underline{\underline{H}}]$$

$$\Rightarrow \frac{d\hat{\underline{\underline{T}}}}{d\underline{\underline{F}}}(\underline{\underline{I}})[\underline{\underline{H}}] = \underset{\sim}{\mathbb{C}}(\underline{\underline{I}})[\underline{\underline{H}}] \quad \forall\, \underline{\underline{H}} \in L(E^3)$$

<u>Corollary</u>

$$\underline{\underline{T}} = \cancel{\hat{\underline{\underline{T}}}(\underline{\underline{I}})}^{\underline{\underline{0}}} + \underset{\sim}{\mathbb{C}}(\underline{\underline{I}})[\underline{\underline{H}}] + o(|\underline{\underline{H}}|)$$

$$\underline{\underline{S}} = \cancel{\hat{\underline{\underline{S}}}(\underline{\underline{I}})}^{\underline{\underline{0}}} + \underset{\sim}{\mathbb{C}}(\underline{\underline{I}})[\underline{\underline{H}}] + o(|\underline{\underline{H}}|)$$

$$\Rightarrow \quad \underline{\underline{S}} = \underline{\underline{T}} + o(|\underline{\underline{H}}|) \text{ as } |\underline{\underline{H}}| \to 0.$$

Exercise (17) Recall (pp. 55-56) that

$$\underline{\underline{S}} = \frac{dW}{d\underline{\underline{F}}}(\underline{\underline{F}}) = \underline{\underline{F}}\,\underline{\underline{P}} = \underline{\underline{F}}\,\frac{d\Psi}{d\underline{\underline{E}}}(\underline{\underline{E}}) .$$

(a) From this identity, show that

$$\underset{\sim}{C}(\underline{\underline{F}})[\underline{\underline{A}}] = \underline{\underline{A}}\,\frac{d\Psi}{d\underline{\underline{E}}}(\underline{\underline{E}}) + \underline{\underline{F}}\,\frac{d^2\Psi}{d\underline{\underline{E}}^2}(\underline{\underline{E}})\left[\tfrac{1}{2}(\underline{\underline{F}}^T\underline{\underline{A}} + \underline{\underline{A}}^T\underline{\underline{F}})\right] \quad \forall\, \underline{\underline{A}} \in L(\mathbb{E}^3),$$

and thus conclude that, in general,

$$\underset{\sim}{C}(\underline{\underline{F}}) \neq \frac{d^2\Psi}{d\underline{\underline{E}}^2}(\underline{\underline{E}}) .$$

(b) On the other hand, assuming a stress-free reference configuration, show that

$$\underset{\sim}{C}(\underline{\underline{I}}) = \frac{d^2\Psi}{d\underline{\underline{E}}^2}(\underline{\underline{O}}) .$$

<u>Pf of Thm (p. 100)</u>

$$\underset{\sim}{\mathbb{C}}(\underline{\underline{I}})[\underline{\underline{H}}] = \frac{d}{d\alpha} \hat{\underline{\underline{T}}}(\underline{\underline{I}} + \alpha \underline{\underline{H}}) \Big|_{\alpha=0}$$

$$= \lim_{\alpha \to 0} \frac{\hat{\underline{\underline{T}}}(\underline{\underline{I}} + \alpha \underline{\underline{H}}) - \cancel{\hat{\underline{\underline{T}}}(\underline{\underline{I}})}^{\underline{\underline{0}}}}{\alpha} \in \mathrm{Sym}(\mathbb{E}^3).$$

For the 2^{nd} part use (MO):

$$\hat{\underline{\underline{S}}}(\underline{\underline{Q}}\,\underline{\underline{F}}) = \underline{\underline{Q}}\,\hat{\underline{\underline{S}}}(\underline{\underline{F}}) \qquad \forall\, \underline{\underline{Q}} \in SO(3).$$

Choose $\underline{\underline{F}} = \underline{\underline{I}}$ and $\underline{\underline{Q}} = \exp(\underline{\underline{W}}t)$ for $\underline{\underline{W}} \in \mathrm{Skew}(\mathbb{E}^3)$ (see pp. 44-45).

Then $$\hat{\underline{\underline{S}}}(\exp(\underline{\underline{W}}t)) = \exp(\underline{\underline{W}}t)\,\cancel{\hat{\underline{\underline{S}}}(\underline{\underline{I}})}^{\underline{\underline{0}}}$$

$$\Rightarrow \frac{d}{dt} \hat{\underline{\underline{S}}}(\exp(\underline{\underline{W}}t)) = \underline{\underline{0}} \quad \text{- in part. at } t=0$$

$$\Rightarrow \quad \underset{\sim}{\mathbb{C}}(\underline{\underline{I}})[\underline{\underline{W}}] = \underline{\underline{0}} \qquad \forall\, \underline{\underline{W}} \in \mathrm{Skew}(\mathbb{E}^3).$$

A reasonable restriction, for which there is ample experimental evidence is:

$$(*) \quad \underline{\underline{H}} \cdot \underset{\sim}{\mathbb{C}}(\underline{\underline{I}})[\underline{\underline{H}}] > 0 \qquad \forall\, \underline{\underline{H}} \in \mathrm{Sym}(\mathbb{E}^3),\ \underline{\underline{H}} \neq \underline{\underline{0}}$$

pos-def'n on symmetric tensor at ref config.

This property (along with smoothness assumptions smoothness of the boundary $\partial\mathcal{B} = \partial\mathcal{B}_1 \cup \partial\mathcal{B}_2$ with $\partial\mathcal{B}_1 \cap \partial\mathcal{B}_2 = \emptyset$), yields a "local" existence result:

First re-write the equilibrium equations in terms of the displacement $\underset{\sim}{X} + \underset{\sim}{u}(\underset{\sim}{X}) = \underset{\sim}{f}(\underset{\sim}{X})$

$\Rightarrow \quad \underset{=}{F} = \underset{=}{I} + \nabla \underset{\sim}{u}, \quad \nabla^2 \underset{\sim}{f} = \nabla^2 \underset{\sim}{u}$:

$$(*) \quad F(\mu, \underset{\sim}{u}) \equiv \begin{cases} \underset{\sim}{\mathbb{C}}(\underset{=}{I} + \nabla \underset{\sim}{u})\, \nabla^2 \underset{\sim}{u} + \hat{\underset{\sim}{b}} = \underset{\sim}{0} & \text{in } \mathcal{B} \\ \underset{\sim}{u} - \hat{\underset{\sim}{d}} = \underset{\sim}{0} & \text{on } \partial\mathcal{B}_1 \quad (\hat{\underset{\sim}{d}} \equiv \hat{\underset{\sim}{f}} - \underset{\sim}{X}) \\ \dfrac{dW}{d\underset{=}{F}}(\underset{=}{I} + \nabla \underset{\sim}{u})\, \underset{\sim}{m} - \hat{\underset{\sim}{s}} = \underset{\sim}{0} & \text{on } \partial\mathcal{B}_2 \end{cases}$$

$\mu \equiv (\hat{\underset{\sim}{b}}, \hat{\underset{\sim}{d}}, \hat{\underset{\sim}{s}})$ prescribed fields.

Solve $\quad F(\mu, \underset{\sim}{u}) = 0 \qquad F : \mathcal{P} \times \mathcal{X} \to \mathcal{Y}$

$\mathcal{P}, \mathcal{X}, \mathcal{Y}$ Banach spaces

<u>Observe</u>: $\quad F(0, \underset{\sim}{0}) = 0$

$\underset{\sim}{\mathbb{C}}(\underset{=}{I})\, \underset{\approx}{0} + \hat{\underset{\sim}{0}} = \underset{\sim}{0}$ ✓

$\underset{\sim}{0} - \underset{\sim}{0} = \underset{\sim}{0}$ ✓

$\cancel{\dfrac{dW}{d\underset{=}{F}}(\underset{=}{I})}\, \underset{\sim}{m} - \underset{\sim}{0} = \underset{\sim}{0}$ ✓