This property (along with smoothness assumption: smoothness of the boundary $\partial B = \partial B_1 \cup \partial B_2$ with $\partial B_1 \cap \partial B_2 = \emptyset$), yields a "local" existence result:

First re-write the equilibrium equations in terms of the displacement $\underline{X} + \underline{u}(\underline{X}) = \underline{f}(\underline{X})$
$\Rightarrow \quad \underline{\underline{F}} = \underline{\underline{I}} + \nabla \underline{u}, \quad \nabla^2 \underline{f} = \nabla^2 \underline{u}$:

$$(*) \quad F(\mu, \underline{u}) \equiv \begin{cases} \underline{\mathbb{C}}(\underline{\underline{I}} + \nabla \underline{u})[\nabla^2 \underline{u}] + \hat{\underline{b}} = \underline{0} & \text{in } B \\ \underline{u} - \hat{\underline{d}} = \underline{0} \text{ on } \partial B_1 & (\hat{\underline{d}} \equiv \hat{\underline{f}} - \underline{X}) \\ \dfrac{d W}{d \underline{\underline{F}}}(\underline{\underline{I}} + \nabla \underline{u})\, \underline{m} - \hat{\underline{s}} = \underline{0} & \text{on } \partial B_2 \end{cases}$$

$\mu \equiv (\hat{\underline{b}}, \hat{\underline{d}}, \hat{\underline{s}})$ prescribed fields.

Solve $\quad F(\mu, \underline{u}) = 0 \qquad F : \mathcal{P} \times \mathcal{X} \to \mathcal{Y}$

$\mathcal{P}, \mathcal{X}, \mathcal{Y}$ (approp. Banach spaces)

<u>Observe</u>: $\quad F(0, \underline{0}) = 0$

$$\underline{\mathbb{C}}(\underline{\underline{I}})\, \underline{\underline{\underline{0}}} + \hat{\underline{0}} = \underline{0} \; \checkmark$$

$$\underline{0} - \underline{0} = \underline{0} \; \checkmark$$

$$\cancel{\frac{dW}{dF}(\underline{\underline{I}})}\, \overset{\underline{\underline{0}}}{\underline{m}} - \underline{0} = \underline{0} \; \checkmark$$

Recall implicit function theorem for finite-dimensional maps:

$$f(\mu, x) = 0 \qquad f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$$

Assume: i) f is C^1

ii) $f(0,0) = 0$ ((0,0) is a known solution)

iii) $\frac{\partial f}{\partial x}(0,0) \in L(\mathbb{R}^n)$ is invertible.

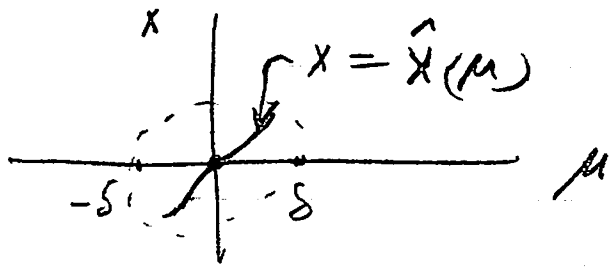
Then $x = \hat{x}(\mu)$ such that

(a) $f(\mu, \hat{x}(\mu)) \equiv 0 \quad \forall \; |\mu| < \delta$

(b) $\hat{x}(0) = 0$

(c) $\hat{x}$ is C^1

(d) "(a)" captures all local solutions



We can carry out the same procedure (formally) for (*) p. 103 :

We must <u>linearize</u> (in $\mathbf{u}$) about the $(\mu, \mathbf{u}) = (0, \mathbf{0})$ solution:

$$\frac{d}{d\alpha} F(0, \mathbf{0} + \alpha \mathbf{h})\Big|_{\alpha=0} \equiv D_u F(0, \mathbf{0})[\mathbf{h}] \quad \forall\, \mathbf{h} \in \mathcal{X}$$

$$= \begin{cases} \dfrac{d}{d\alpha}\left[\mathbb{C}(\mathbf{I} + \alpha \nabla \mathbf{h})(\alpha \nabla^2 \mathbf{h}) + \mathbf{0}\right]\Big|_{\alpha=0} \\ \dfrac{d}{d\alpha}\left[(\alpha \mathbf{h}) - \mathbf{0}\right]\Big|_{\alpha=0} \\ \dfrac{d}{d\alpha}\left[\dfrac{dW}{d\mathbf{F}}(\mathbf{I} + \alpha \nabla \mathbf{h})\mathbf{m} - \mathbf{0}\right]\Big|_{\alpha=0} \end{cases}$$

$$= \begin{cases} \mathbb{C}(\mathbf{I})[\nabla^2 \mathbf{h}] & \text{in } B \\ \mathbf{h} & \text{on } \partial B_1 \\ \mathbb{C}(\mathbf{I})[\nabla \mathbf{h}]\mathbf{m} & \text{on } \partial B_2 \end{cases}$$

Is the linear transformation $D_u F(0, \mathbf{0}) : \mathcal{X} \to \mathcal{Y}$ invertible?

<u>1-1</u>: $D_u F(0, \mathbf{0})[\mathbf{h}] = 0$ have a unique solution?,

i.e., does

$$\begin{cases} \mathbb{C}(\mathbf{I})\, \nabla^2 \mathbf{h} = \mathbf{0} & \text{in } B \\ \mathbf{h} = \mathbf{0} & \text{on } \partial B_1 \\ \mathbb{C}(\mathbf{I})[\nabla \mathbf{h}]\mathbf{m} = \mathbf{0} & \text{on } \partial B_2 \end{cases}$$

have the <u>unique</u> solution $\underset{\sim}{h} \equiv \underset{\sim}{0}$?

If so, $D_h F(0, \underset{\sim}{0})$ is 1-1.

<u>Ans</u>: yes

<u>Pf</u>: Kirchhoff's uniqueness theorem for linear elasticity. Observe

$$\underline{\mathbb{C}}(\underline{\underline{I}})\, \underline{\nabla}^2 \underline{h} = \underset{\sim}{\nabla} \cdot \left(\underline{\mathbb{C}}(\underline{\underline{I}}) [\underline{\nabla} \underline{h}] \right)$$

"Mult." by $\underset{\sim}{h}$ and integrate over $\mathcal{B}$:

$$\int_{\mathcal{B}} \underset{\sim}{h} \cdot \underset{\sim}{\nabla} \cdot \left(\underline{\mathbb{C}}(\underline{\underline{I}}) (\underline{\nabla} \underline{h}] \right) dV$$

Again, from p. 35: $(\underline{\nabla} \cdot \underline{\underline{S}}) \cdot \underset{\sim}{n} = \underline{\nabla} \cdot (\underline{\underline{S}}^T \underset{\sim}{n}) - \underline{\underline{S}} \cdot \nabla \underset{\sim}{n}$

"$\underline{\underline{S}}$" $\equiv \underline{\mathbb{C}}(\underline{\underline{I}})[\nabla \underline{h}]$ ↓

$$\overset{\text{div thm}}{=} -\int_{\mathcal{B}} \underset{\sim}{\nabla} \underline{h} \cdot \underline{\mathbb{C}}(\underline{\underline{I}}) [\underline{\nabla} \underline{h}]\, dV \qquad \text{"}\underline{\underline{S}}^T\text{"} = \underline{\underline{S}}$$

$$+ \int_{\partial \mathcal{B}_2} \cancel{\underline{\mathbb{C}}(\underline{\underline{I}})[\underline{\nabla} \underline{h}]\, \underset{\sim}{m} \cdot \underset{\sim}{h}} \to \underset{\sim}{0} \text{ b.c.}$$

$$\therefore \int_{\mathcal{B}} \nabla \underline{h} \cdot \underline{\mathbb{C}}(\underline{\underline{I}}) [\underline{\nabla} \underline{h}]\, dV = 0$$

Recall $\mathbb{C}(\underline{\underline{I}})[\underline{\underline{W}}] = \underline{\underline{0}} \ \forall \ \underline{\underline{W}} \in$ skew.

For a given $\nabla \underline{u}$, write $\nabla \underline{u} = \underline{\underline{H}} + \underline{\underline{W}}$, where $\underline{\underline{H}} = \frac{1}{2}(\nabla \underline{u} + \nabla \underline{u}^T)$ and $\underline{\underline{W}} = \frac{1}{2}(\nabla \underline{u} - \nabla \underline{u}^T)$.

Then $(\underline{\underline{H}} + \underline{\underline{W}}) \cdot \mathbb{C}(\underline{\underline{I}})[\underline{\underline{H}} + \underline{\underline{W}}]$

$$= (\underline{\underline{H}} + \underline{\underline{W}}) \cdot \underset{\sim}{\mathbb{C}}(\underline{\underline{I}})[\underline{\underline{H}}] \qquad (\mathbb{C}(\underline{\underline{I}})[\underline{\underline{W}}] = \underline{\underline{0}})$$

$$= \underbrace{\underline{\underline{H}} \cdot \underset{\sim}{\mathbb{C}}(\underline{\underline{I}})[\underline{\underline{H}}]}_{> 0 \ \forall\, \underline{\underline{H}} \in Sym(\mathbb{E}^3),\ \underline{\underline{H}} \neq \underline{\underline{0}}.} + \underset{\sim}{\mathbb{C}}(\underline{\underline{I}})\underset{\underline{\underline{0}}}{\cancel{[\underline{\underline{W}}]}} \cdot \underline{\underline{H}} \quad \begin{pmatrix}\text{Lemma} \\ \text{p. 99}\end{pmatrix}$$

Thus we conclude that $\nabla \underline{u}(\underline{X}) \in Skew(\mathbb{E}^3)$. Now if $\underset{\sim}{u}$ is C^2 (which we assume), then

$$\nabla \underline{u} = -\nabla \underline{u}^T$$

$$\Rightarrow \frac{\partial u_i}{\partial X_j} = -\frac{\partial u_j}{\partial X_i}$$

Then $\dfrac{\partial^2 u_i}{\partial X_k \partial X_j} = -\dfrac{\partial^2 u_j}{\partial X_k \partial X_i}$

$$\Rightarrow \frac{\partial}{\partial X_j}\left(\frac{\partial u_i}{\partial X_k}\right) = -\frac{\partial}{\partial X_i}\left(\frac{\partial u_j}{\partial X_k}\right)$$

$$\Rightarrow -\frac{\partial}{\partial X_j}\left(\frac{\partial \overset{\downarrow}{u_k}}{\partial X_i}\right) = \frac{\partial}{\partial X_i}\left(\frac{\partial \overset{\downarrow}{u_k}}{\partial X_j}\right) \Rightarrow \frac{\partial^2 u_k}{\partial X_i \partial X_j} \equiv 0$$

$$\Rightarrow \nabla(\nabla \underline{u}) \equiv \underline{\underline{0}}$$

$$\Rightarrow \nabla u(\underline{X}) \equiv W_0 \in Skew$$

$$\Rightarrow \quad \underset{\sim}{u} = \underline{\underline{W}}_0 \underline{X} + \underline{c}$$

$$\underset{\sim}{u}\Big|_{\partial B_1} = \underset{\sim}{0} \quad \Rightarrow \quad \underline{\underline{W}}_0 = \underline{\underline{0}}, \ \underline{c} = \underline{0} .$$

To show that $D_u F(0, \underset{\sim}{0})$ is onto, we would need to demonstate that any inhomogeneous problem $D_u F(0, \underset{\sim}{0})[\underset{\sim}{h}] = \hat{g}$ has a solution, viz.,)

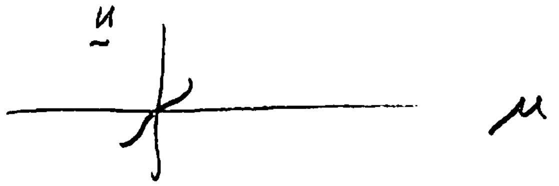
$$\begin{cases} \mathbb{C}(\underline{\underline{I}})\, \underline{\nabla}^2 \underline{h} = \underset{\sim}{\hat{g}}_1 & \text{in } \mathcal{B}, \\ \underset{\sim}{h} = \underset{\sim}{\hat{g}}_2 & \text{on } \partial \mathcal{B}_1, \\ \mathbb{C}(\underline{\underline{I}})[\underline{\nabla}\underline{h}]\underset{\sim}{m} = \underset{\sim}{\hat{g}}_3 & \text{on } \partial \mathcal{B}_2, \end{cases}$$

has a solution "$\underset{\sim}{h}$". This is much harder - but doable - using elliptic pde theory (which we must skip here).

With this in hand, we then have a "path" of solutions

$$\underset{\sim}{u}(\underline{X}) = \underset{\sim}{\hat{u}}\Big(\underline{X}\, ; \overbrace{(\underset{\sim}{\hat{b}}, \underset{\sim}{\hat{d}}, \underset{\sim}{\hat{s}})}^{\mu}\Big)$$

for "$|\mu|$ sufficiently small", to the nonlinear problem.



To motivate our next hypothesis for $\mathbb{C}(\mathbf{F})$, we suppose that $\mathbf{f}_0 = \mathbf{X} + \mathbf{u}_0$ is an equilibrium solution, viz.,

$$\mathbb{C}(\mathbf{I}+\nabla\mathbf{u}_0)[\nabla^2\mathbf{u}_0] + \hat{\mathbf{b}} \equiv \mathbf{0} \quad \text{in } \mathcal{B},$$

$$\mathbf{u}_0 \equiv \hat{\mathbf{d}} \quad \text{on } \partial\mathcal{B}_1,$$

$$\frac{dW}{d\mathbf{F}}(\nabla\mathbf{f}_0)\,\mathbf{m} \equiv \hat{\mathbf{s}} \quad \text{on } \partial\mathcal{B}_2,$$

and we linearize the <u>dynamical</u> equations (p. 77) at $\mathbf{u}_0$:

$$\mathbf{f}(\mathbf{X},t) = \overset{\mathbf{X}+\mathbf{u}_0(\mathbf{X})}{\mathbf{f}_0(\mathbf{X})} + \alpha\,\mathbf{w}(\mathbf{X},t)$$

Plug into dynamical equations and linearize:

$$\frac{d}{d\alpha}\Big[\mathbb{C}(\mathbf{F}_0 + \alpha\nabla\mathbf{w})[\nabla^2\mathbf{f}_0 + \alpha\nabla^2\mathbf{w}] + \hat{\mathbf{b}} = \rho_0(\alpha\,\mathbf{w}_{tt})\Big]\Big|_{\alpha=0}$$

$$\frac{d}{d\alpha}\left(\mathbf{f}_0 + \alpha\mathbf{w} - \hat{\mathbf{f}}\right)\Big|_{\alpha=0} = \mathbf{0}$$

$$\frac{d}{d\alpha}\left(\frac{dW}{d\mathbf{F}}(\mathbf{F}_0 + \alpha\nabla\mathbf{w})\,\mathbf{m} - \hat{\mathbf{s}}\right)\Big|_{\alpha=0} = \mathbf{0},$$

which leads to:

$$\left\{\begin{aligned} &\underset{\sim}{\mathbb{C}}(\underline{\underline{F}}_0)[\underset{\sim}{\nabla}^2 \underset{\sim}{w}] + \overbrace{\frac{d\mathbb{C}}{d\underline{\underline{F}}}(\underline{\underline{F}}_0)[\underset{\sim}{\nabla} \underset{\sim}{w}]\, \underset{\sim}{\nabla}^2 \underset{\sim}{f}_0}^{\text{lower order}} \\ &\qquad\qquad = \rho_0\, \underset{\sim}{w}_{tt} \qquad \text{in } \mathcal{B} \\ &\underset{\sim}{w} = \underset{\sim}{0} \qquad \text{on } \partial\mathcal{B}_1 \\ &\underset{\sim}{\mathbb{C}}(\underline{\underline{F}}_0)[\underset{\sim}{\nabla} \underset{\sim}{w}]\, \underset{\sim}{m} = \underset{\sim}{0} \qquad \text{on } \partial\mathcal{B}_2 \end{aligned}\right.$$

Linearized dynamical equations about an arbitrary equilibrium. Let's make the assumption that $\underset{\sim}{f}_0$ is homogeneous, i.e., $\underline{\underline{F}}_0 = \underline{\underline{\text{Const}}}$. Then the pde becomes:

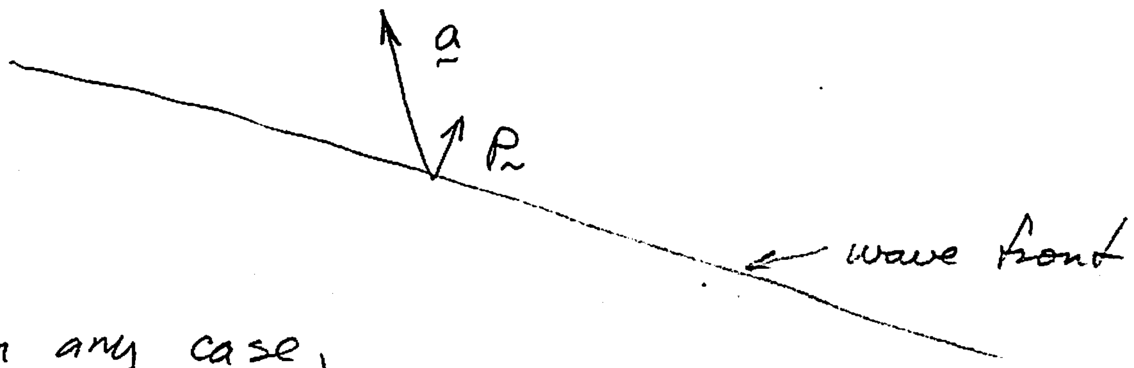
$$(*) \quad \underset{\sim}{\nabla}\cdot\left(\underset{\sim}{\mathbb{C}}(\underline{\underline{F}}_0)[\underset{\sim}{\nabla}\underset{\sim}{w}]\right) = \rho_0\, \underset{\sim}{w}_{tt}$$

Let's also ignore the b.c.'s (e.g. infinite medium) and use wave propogation to guide us in a proper assumption for $\underset{\sim}{\mathbb{C}}(\underline{\underline{F}}_0)$ (recall, pos-def'n on sym. tensors is not good for $\underline{\underline{F}}_0 \neq \underline{\underline{I}}$, pp. 94-96)

Seek a planar travelling wave solution of $(*)$:

$$\underset{\sim}{w}(\underset{\sim}{X}, t) = \underset{\sim}{a}\; g(\underset{\sim}{p}\cdot\underset{\sim}{X} - ct) \;,$$

where $\underset{\sim}{p}$ is a unit vector – ⊥ to the wave front; $\underset{\sim}{p}$ is called the <u>direction of propogation</u>, and $\underset{\sim}{a}$ is the <u>direction of motion</u>:



In any case,

$$\underset{\sim}{\nabla}\underset{\sim}{w}\;\underset{\sim}{\eta} = \frac{d}{d\alpha}\,\underset{\sim}{a}\,g\left(\underset{\sim}{p}\cdot(\underset{\sim}{X}+\alpha\underset{\sim}{\eta})-ct\right)\Big|_{\alpha=0}$$

$$= \underset{\sim}{a}\,g'(\underset{\sim}{p}\cdot\underset{\sim}{X}-ct)\,\underset{\sim}{p}\cdot\underset{\wedge}{\eta}$$

$$= \underbrace{\left[\underset{\sim}{a}\otimes\underset{\sim}{p}\;g'(\underset{\sim}{p}\cdot\underset{\sim}{X}-ct)\right]}_{\underset{\sim}{\nabla}\underset{\sim}{w}}\underset{\sim}{\eta}\quad\forall\;\underset{\sim}{\eta}\in\mathbb{E}^3$$

Then $\left(\underset{\sim}{\nabla}\cdot\left(\underset{\sim}{C}(\underset{=}{F}_o)[\underset{\sim}{\nabla}\underset{\sim}{w}]\right)\right)_i$

$$= \left(\underset{\sim}{\nabla}\cdot\left(\underbrace{\underset{\sim}{C}(\underset{=}{F}_o)[\underset{\sim}{a}\otimes\underset{\sim}{p}]}_{\underset{=}{B}^{\circ}}\,g'(\underset{\sim}{p}\cdot\underset{\sim}{X}-ct)\right)\right)_i$$

$$= \frac{d}{dX_j}\left(B^{\circ}_{ij}\;g'(p_k X_k - ct)\right)$$

$$= B^{\circ}_{ij}\;g''(p_k X_k - ct)\,\overset{p_j}{\cancel{p_k}\,\delta_{kj}}$$

$$\underset{\sim}{\nabla}\cdot\left(\underset{\sim}{\mathbb{C}}(\underset{=}{F}_0)[\nabla \underset{\sim}{w}]\right)$$

$$= g''(\cdot)\ \underset{\sim}{\mathbb{C}}(\underset{=}{F}_0)[\underset{\sim}{a}\otimes\underset{\sim}{p}]\,\underset{\sim}{p}$$

$\therefore\ (*) \Rightarrow$

$$g''(\cdot)\ \underset{\sim}{\mathbb{C}}(\underset{=}{F}_0)[\underset{\sim}{a}\otimes\underset{\sim}{p}]\,\underset{\sim}{p} = \rho_0 c^2 \underset{\sim}{a}\, g''(\cdot)$$

(assuming $g''(\cdot) \neq 0$)

$$\Rightarrow \boxed{\underset{\sim}{\mathbb{C}}(\underset{=}{F}_0)[\underset{\sim}{a}\otimes\underset{\sim}{p}]\,\underset{\sim}{p} = \rho_0 c^2 \underset{\sim}{a}} \qquad (*)$$

question: for fixed $\underset{\sim}{p}$, why does this define a tensor?

Defn $\underset{=}{Q}(\underset{\sim}{p})\,\underset{\sim}{a} \equiv \underset{\sim}{\mathbb{C}}(\underset{=}{F}_0)[\underset{\sim}{a}\otimes\underset{\sim}{p}]\,\underset{\sim}{p}$, where

$\underset{=}{Q}(\underset{\sim}{p})$ is called the acoustic tensor in the direction of $\underset{\sim}{p}$.

For a given $\underset{\sim}{p}$, $(*)$ above is an eigenvalued problem:

$$\boxed{\underset{=}{Q}(\underset{\sim}{p})\,\underset{\sim}{a} = \rho_0 c^2 \underset{\sim}{a}} \qquad (**)$$

Observe: $\underset{\sim}{b}\cdot\underset{=}{Q}(\underset{\sim}{p})\,\underset{\sim}{a} = \underset{\sim}{b}\cdot\underset{\sim}{\mathbb{C}}(\underset{=}{F}_0)[\underset{\sim}{a}\otimes\underset{\sim}{p}]\,\underset{\sim}{p}$

$$= b_i\ \overset{\circ}{\mathbb{C}}_{ijkl}\, a_k\, p_l\, p_j$$

$$= b_i\, p_j\ \overset{\circ}{\mathbb{C}}_{ijkl}\, a_k\, p_l$$

$$= \underset{\sim}{b}\otimes\underset{\sim}{p}\cdot\underset{\sim}{\mathbb{C}}(\underset{=}{F}_0)[\underset{\sim}{a}\otimes\underset{\sim}{p}]$$

Lemma p. 99

$$= \underset{\sim}{a} \otimes \underset{\sim}{p} \cdot \underset{\sim}{C}(\underset{=}{F}_0)[\underset{\sim}{b} \otimes \underset{\sim}{p}]$$

$$= \underset{\sim}{a} \cdot \underset{\approx}{Q}(\underset{\sim}{p})\, \underset{\sim}{b} \,,$$

i.e., $\underset{=}{Q}^T = \underset{=}{Q}$. So the eigenvalues $\sigma = \rho_0 c^2$ of $\underset{=}{Q}$ are real. We would also like them to be positive (otherwise we could hav imaginary wave speeds)

<u>Require</u> $\underset{\sim}{a} \cdot \underset{=}{Q}(\underset{\sim}{p})\, \underset{\sim}{a} > 0$

or (SE) $\boxed{\underset{\sim}{a} \otimes \underset{\sim}{p} \cdot \underset{\sim}{C}(\underset{=}{F}_0)[\underset{\sim}{a} \otimes \underset{\sim}{p}] > 0 \quad \forall\, \underset{\sim}{a}, \underset{\sim}{p} \neq \underset{\sim}{0}.}$

This is called <u>strong</u> <u>ellipticity</u> at $\underset{=}{F}_0$. Insures well-posed "incremental" dynamics.

<u>Exercise</u> ⑱ For an isotropic solid we have

$$\underset{=}{C}(\underset{=}{I})[\underset{=}{H}] = \Lambda\, (\mathrm{tr}\, \underset{=}{H})\, \underset{=}{I} + 2\mu\, \underset{=}{H} \qquad \forall\, \underset{=}{H} \in \mathrm{Sym}(\mathbb{E}^3$$

where Λ, μ are the Lamé constants (see any book on linear elasticity). (a) Show that the acoustic tensor (at the reference configuration) has the form

$$\underset{=}{Q}(\underset{\sim}{p}) = (\Lambda + 2\mu)\, \underset{\sim}{p} \otimes \underset{\sim}{p} + \mu\, (\underset{=}{I} - \underset{\sim}{p} \otimes \underset{\sim}{p})$$

(b) Deduce: strong ellipticity of $\underset{=}{C}(\underset{=}{I}) \Leftrightarrow$

$$\Lambda + 2\mu > 0 \quad \text{and} \quad \mu > 0$$

(c) For a given direction of propogation "$\underset{\sim}{p}$", compute the possible wave speeds "c" (from the eigenvalues) and corresponding directions of motion "$\underset{\sim}{a}$" (eigenvectors)

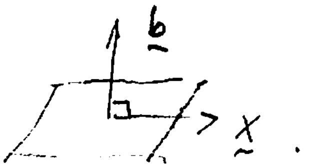
Note that (SE) is weaker than positive definiteness, which reads

$$(*) \quad \underline{\underline{A}} \cdot \underset{\sim}{\mathbb{C}}(\underline{\underline{F}}_0)[\underline{\underline{A}}] > 0 \qquad \forall\ \underset{\underline{\underline{A}} \neq \underline{\underline{0}}}{\underline{\underline{A}} \in L(\mathbb{E}^3)}.$$

Indeed, whereas (*) above is valid for *any* tensor $\underline{\underline{A}} \neq \underline{\underline{0}}$, (SE) involves only tensors of the form $\underset{\sim}{a} \otimes \underset{\sim}{b}$, $\underset{\sim}{a}, \underset{\sim}{b} \neq \underset{\sim}{0}$, which always have rank $= 1$:

$$(\underset{\sim}{a} \otimes \underset{\sim}{b})\underset{\sim}{x} \equiv \underset{\sim}{a}(\underset{\sim}{b} \cdot \underset{\sim}{x}) = \underset{\sim}{0} \qquad \forall\ \underset{\sim}{x} \perp \underset{\sim}{b},$$

i.e., $\underset{\sim}{a} \otimes \underset{\sim}{b}$ has a 2-dimensional null space (the plane $\perp$ to $\underset{\sim}{b}$):

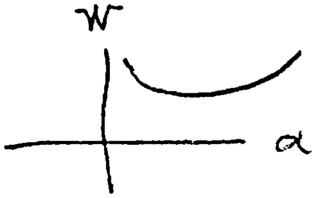


In fact, (*) $\Rightarrow$ convexity, which as we know is not good physics. To see this, consider the function

$$\alpha \mapsto W(\underline{\underline{F}}_0 + \alpha \underline{\underline{A}}) \qquad (\underline{\underline{F}}_0, \underline{\underline{A}} \neq \underline{\underline{0}} \text{ fixed})$$

Then $$\frac{d^2}{d\alpha^2} W(\underline{\underline{F}}_0 + \alpha \underline{\underline{A}}) = \frac{d}{d\alpha}\left(\frac{dW}{d\underline{\underline{F}}}(\underline{\underline{F}}_0 + \alpha\underline{\underline{A}}) \cdot \underline{\underline{A}} \right)$$

positive 2nd derivative!



$$= \underline{\underline{A}} \cdot \frac{d^2 W}{d\underline{\underline{F}}^2}(\underline{\underline{F}}_0 + \alpha\underline{\underline{A}})[\underline{\underline{A}}] > 0$$

from (*),

$\Rightarrow$ $\alpha \mapsto W(\underline{\underline{F}}_0 + \alpha\underline{\underline{A}})$ is convex – which is (can be shown) equivalent to convexity of $W(\underline{\underline{F}})$ (p. 93).

On the other hand, (SE) also implies a kind of convexity, but only in "rank-one" directions, i.e., only "rank-one excursions" from a given $\underline{\underline{F}}_0$ are allowed: $\underline{\underline{F}}_0 + \alpha\, \underset{\sim}{a} \otimes \underset{\sim}{b}$

$\underset{\sim}{a}, \underset{\sim}{b} \neq \underset{\sim}{0}$

Then $$\left.\frac{d}{d\alpha} W(\underline{\underline{F}}_0 + \alpha\, \underset{\sim}{a} \otimes \underset{\sim}{b})\right|_{\alpha=0} = \underset{\sim}{a} \otimes \underset{\sim}{b} \cdot \frac{dW}{d\underline{\underline{F}}}(\underline{\underline{F}}_0)$$

$$\equiv \tilde{S}_{\alpha\beta}(\underline{\underline{F}}_0)$$ comp. 1st Pk

Also $$\left.\frac{d^2}{d\alpha^2} W(\underline{\underline{F}}_0 + \alpha\, \underset{\sim}{a} \otimes \underset{\sim}{b})\right|_{\alpha=0}$$

$$= \left.\frac{d}{d\alpha}\left[\frac{dW}{d\underline{\underline{F}}}(\underline{\underline{F}}_0 + \alpha\, \underset{\sim}{a} \otimes \underset{\sim}{b}) \cdot \underset{\sim}{a} \otimes \underset{\sim}{b} \right]\right|_{\alpha=0}$$

$$\equiv \frac{d\tilde{S}_{\alpha\beta}}{dF_{\alpha\beta}}(\underline{\underline{F}}_0) = \underset{\sim}{a} \otimes \underset{\sim}{b} \cdot \underset{\sim}{C}(\underline{\underline{F}}_0)[\underset{\sim}{a} \otimes \underset{\sim}{b}] > 0$$