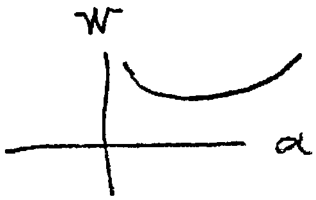
Lecture 14

Then $\dfrac{d^2}{d\alpha^2} W(\underline{\underline{F}}_0 + \alpha \underline{\underline{A}}) = \dfrac{d}{d\alpha}\left(\dfrac{dW}{d\underline{\underline{F}}}(\underline{\underline{F}}_0 + \alpha \underline{\underline{A}}) \cdot \underline{\underline{A}} \right)$

positive 2nd derivative!



$$= \underline{\underline{A}} \cdot \frac{d^2W}{d\underline{\underline{F}}^2}(\underline{\underline{F}}_0 + \alpha \underline{\underline{A}})[\underline{\underline{A}}] > 0$$

from (*),

$\Rightarrow \quad \alpha \mapsto W(\underline{\underline{F}}_0 + \alpha \underline{\underline{A}})$ is convex – which is (can be shown) equivalent to convexity of $W(\underline{\underline{F}})$ (p. 93).

On the other hand, (SE) also implies a kind of convexity, but only in "rank-one" directions, i.e., only "rank-one excursions" from a given $\underline{\underline{F}}_0$ are allowed: $\underline{\underline{F}}_0 + \alpha\, \underset{\sim}{a} \otimes \underset{\sim}{b}$

$\underset{\sim}{a}, \underset{\sim}{b} \neq \underset{\sim}{0}$

Then $\dfrac{d}{d\alpha} W(\underline{\underline{F}}_0 + \alpha\, \underset{\sim}{a} \otimes \underset{\sim}{b})\Big|_{\alpha=0} = \underset{\sim}{a} \otimes \underset{\sim}{b} \cdot \dfrac{dW}{d\underline{\underline{F}}}(\underline{\underline{F}}_0)$

$\equiv \tilde{S}_{\alpha\beta}(\underline{\underline{F}}_0)$ comp. 1st PK

Also $\dfrac{d^2}{d\alpha^2} W(\underline{\underline{F}}_0 + \alpha\, \underset{\sim}{a} \otimes \underset{\sim}{b})\Big|_{\alpha=0}$

$$= \frac{d}{d\alpha}\left[\frac{dW}{d\underline{\underline{F}}}(\underline{\underline{F}}_0 + \alpha\, \underset{\sim}{a} \otimes \underset{\sim}{b}) \cdot \underset{\sim}{a} \otimes \underset{\sim}{b} \right]\Bigg|_{\alpha=0}$$

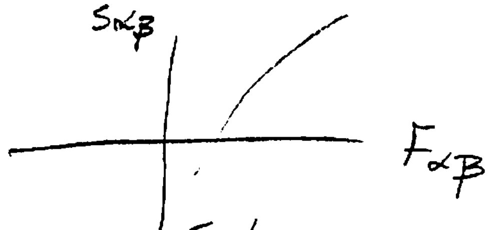
$$\equiv \frac{d\tilde{S}_{\alpha\beta}}{dF_{\alpha\beta}}(\underline{\underline{F}}_0) = \underset{\sim}{a} \otimes \underset{\sim}{b} \cdot \underset{\approx}{C}(\underline{\underline{F}}_0)[\underset{\sim}{a} \otimes \underset{\sim}{b}] > 0$$

This is physically appealing : observe that

$\alpha\, \underset{\sim}{a} \otimes \underset{\sim}{a}$ represents an extension along $\underset{\sim}{a}$, while
$\alpha\, \underset{\sim}{e}_1 \otimes \underset{\sim}{e}_2$ " a shear. (SE) $\Rightarrow$

$$\frac{d \hat{\tilde{S}}_{\alpha\beta}}{d F_{\alpha\beta}}(\underset{=}{F}_0) > 0$$

rank-one
<u>monotone</u> <u>increasing</u>



$$S_{\alpha\beta} \equiv \underset{\sim}{a} \otimes \underset{\sim}{b} \cdot \underset{=}{S} = \underset{\sim}{a} \cdot \underset{=}{S}\, \underset{\sim}{b}$$

$$F_{\alpha\beta} \equiv \underset{\sim}{a} \otimes \underset{\sim}{b} \cdot \underset{=}{F} = \underset{\sim}{a} \cdot \underset{=}{F}\, \underset{\sim}{b}\,.$$

We remark that, as the name suggests, (SE) $\Rightarrow$ static bvp (p.104) is <u>elliptic</u>. In particular, the linearized static problem (p. 111 with $w_t = w_{tt} \equiv 0$) is a well-posed linear elliptic pde — subject to the <u>Fredholm</u> <u>alternative</u>, i.e., non uniqueness is allowed (bifurcation)!

Verifying (SE) for specific material is not an easy task. For the Green-Hadamard material (p.73), it can be shown that $a > 0$, $b \geq 0$ and $\Gamma''(J) \geq 0 \;\; \forall J \Rightarrow$ (SE).

<u>Caution</u> Returning to the results of <u>Exercise</u> (17), suppose that we assume that $\dfrac{\partial^2 \psi}{\partial \underline{\underline{E}}^2}(\underline{\underline{E}})$ is positive definite. Observe that

$$\underset{\sim}{a} \otimes \underset{\sim}{b} \cdot \underline{\underline{C}}(\underline{\underline{F}})[\underset{\sim}{a} \otimes \underset{\sim}{b}]$$

$$= \underset{\sim}{a} \otimes \underset{\sim}{b} \cdot \left(\underset{\sim}{a} \otimes \underset{\sim}{b} \frac{d\psi}{d\underline{\underline{E}}}(\underline{\underline{E}})\right)$$

$$+ \underset{\sim}{a} \otimes \underset{\sim}{b} \cdot \underline{\underline{F}} \frac{d^2\psi}{d\underline{\underline{E}}^2}(\underline{\underline{E}}) \left[\tfrac{1}{2}\left(\underline{\underline{F}}^T(\underset{\sim}{a} \otimes \underset{\sim}{b}) + (\underset{\sim}{b} \otimes \underset{\sim}{a})\underline{\underline{F}}\right)\right]$$

Clearly, we have no "control" over $\dfrac{d\psi}{d\underline{\underline{E}}}(\underline{\underline{E}})$

$\Rightarrow$ ellipticity can fail!

To finish this part on the 3-dimensional ~~theory~~, we return to the calculus of variations to briefly discuss the work of John Ball. Recall the Green-Hadamard material:

$$(*) \quad W(\underline{\underline{F}}) = a\, \underline{\underline{F}} \cdot \underline{\underline{F}} + b \operatorname{Cof} \underline{\underline{F}} \cdot \operatorname{Cof} \underline{\underline{F}} + \Gamma(\det \underline{\underline{F}}),$$

with $a > 0$, $b > 0$ and $\Gamma(\cdot)$ convex.

It turns out that $\underline{\underline{F}} \mapsto \operatorname{Cof} \underline{\underline{F}}$ and $\underline{\underline{F}} \mapsto \det \underline{\underline{F}}$ are not convex functions. Hence, $\underline{\underline{F}} \mapsto W(\underline{\underline{F}})$ above $(*)$ is not convex, which is good. On the other hand, $\underline{\underline{F}} \mapsto \underline{\underline{F}} \cdot \underline{\underline{F}}$,

$\underline{\underline{H}} \mapsto \underline{\underline{H}} \cdot \underline{\underline{H}}$ and $J \mapsto \Gamma(J)$ are each convex functions. Thus,

$$(\underline{\underline{F}}, \underline{\underline{H}}, J) \mapsto a\, \underline{\underline{F}} \cdot \underline{\underline{F}} + b\, \underline{\underline{H}} \cdot \underline{\underline{H}} + \Gamma(J)$$

is convex.

More generally, Ball considered

$$W(\underline{\underline{F}}) = \Phi(\underline{\underline{F}}, \operatorname{Cof} \underline{\underline{F}}, \det \underline{\underline{F}}),$$

with $W \to \infty$ as $|\underline{\underline{F}}| \to \infty$ and/or $\det \underline{\underline{F}} \searrow 0$, where $(\underline{\underline{F}}, \underline{\underline{H}}, J) \mapsto \Phi(\underline{\underline{F}}, \underline{\underline{H}}, J)$ is convex. Such a function is said to be <u>poly-convex</u>. For a <u>very general</u> class of mixed boundary value problems (without assuming smoothness), Ball [1977] was able to show that

$$\mathcal{V}[\underline{f}] = \int_{\mathcal{B}} \Phi(\nabla \underline{f}, \operatorname{Cof} \nabla \underline{f}, \det \nabla \underline{f})\, dV + \text{cons.} \begin{cases} \text{body force} \\ \text{traction} \end{cases}$$

has a minimum $\underset{\sim}{f}^*$ (not necessarily unique) in $W^{1,p}(\mathcal{B})$ satisfying $\det \nabla \underline{f}^* > 0$ a.e..

Moreover, polyconvexity $\Rightarrow$

$$\underset{\sim}{a} \otimes \underset{\sim}{b} \cdot \underset{\sim}{\mathcal{C}}(\underline{\underline{F}})[\underset{\sim}{a} \otimes \underset{\sim}{b}] \geq 0 \quad \forall\, \underset{\sim}{a}, \underset{\sim}{b} \neq \underset{\sim}{0}$$

not quite (SE)

$\Rightarrow$ rank-one convexity.

However, is $\underset{\sim}{f}^*$ a weak solution?

<u>unknown</u> - even for assumed smoothness

Because $\frac{d}{d\alpha} \mathcal{V}(\underset{\sim}{f}^* + \alpha \underset{\sim}{\eta})\big|_{\alpha=0} (= 0)$ cannot be done rigorous

Special Cosset Theory of Rods

Roughly speaking, a rod is a "long, thin body" = L (length) >> area of cross-section. There are two basic approaches:

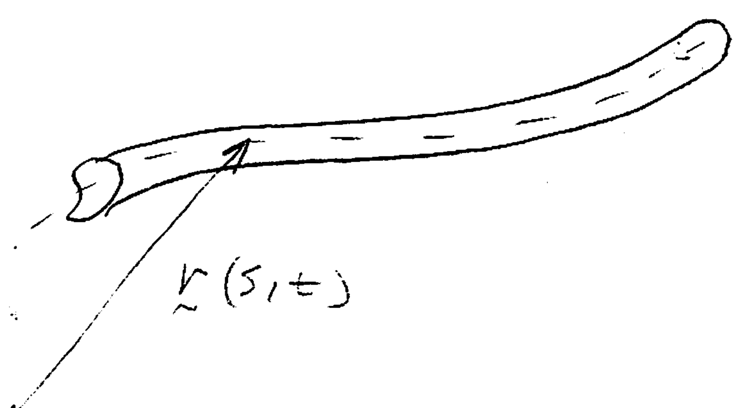
① Direct approach: treat rods as space curves with "micro structure"

② Treat rods as constrained 3-d elastic bodies

We will freely "bounce between" those two approaches – although we will simply employ ② to motivate ① in the end.

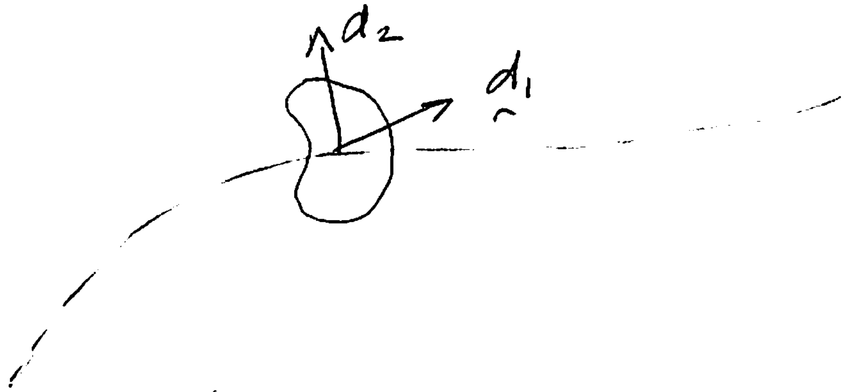
Some Preliminary Kinematics

Let $s \mapsto \underset{\sim}{r}(s,t)$ denote the "centroidal" curve of a "slender" 3-dimensional body:

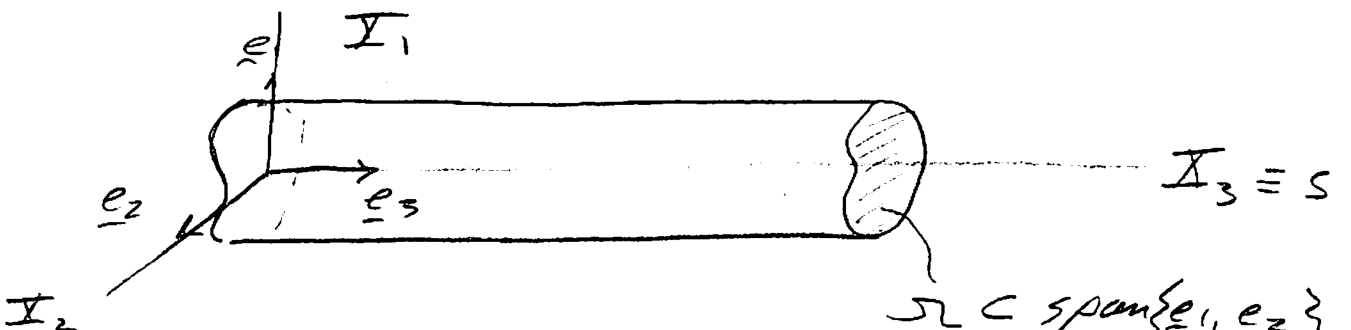


To model the cross-section, we attach two spanning vector fields $\underset{\sim}{d}_1(s,t)$, $\underset{\sim}{d}_2(s,t)$

at the centroid as (s,t):



For simplicity, we assume a straight, ~~prismatic~~ reference configuration:



We choose "s" to be the arclength in the straight reference configuration. Since the $X_3 = s$ axis is centroidal, we have

$$\int_\Omega X_\alpha \, dA = 0 \quad , \quad \alpha = 1, 2.$$

Moreover, we choose e_1 & e_2 to be principal:

$$\int_\Omega X_1 X_2 \, dA = 0 .$$

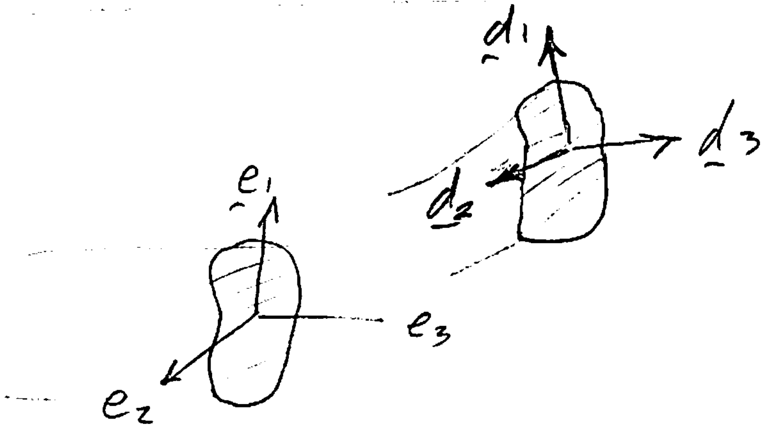
Defn: A deformation of a rod is a

mapping $s \mapsto \underset{\sim}{r}(s), \underset{\sim}{d}_\alpha(s)$, $\alpha = 1, 2$. A <u>motion</u> of a rod is a one-parameter family of deformations $\underset{\sim}{r}(s,t), \underset{\sim}{d}_\alpha(s,t)$.

<u>Motivation</u> plane sections remain plane:

Greek indices sum 1, 2

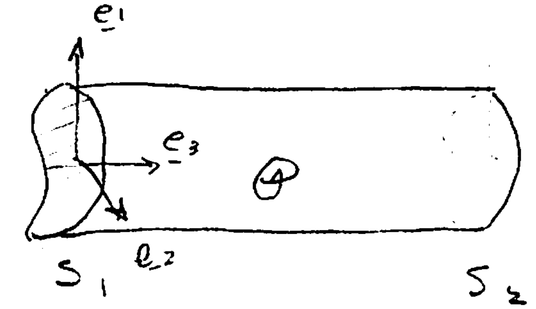
$$\boxed{\underset{\sim}{f}(\underset{\sim}{X}) = \underset{\sim}{r}(s) + X_\alpha \underset{\sim}{d}_\alpha(s)} \qquad (s = X_3)$$



We view $\underset{\sim}{d}_1(s)$ & $\underset{\sim}{d}_2(s)$ as spanning the plane of the deformed image of Ω. The vectors $\underset{\sim}{d}_1(s)$ & $\underset{\sim}{d}_2(s)$ are called <u>directors</u>.

Balance of Linear Momentum

Consider a segment of the straight, prismatic body:



Referring back to p. 28, we have

$$(*) \quad \int_{\partial \mathcal{P}} \underset{\approx}{S}\, \underset{\sim}{m}\, dS + \int_{\mathcal{P}} \hat{\underset{\sim}{b}}\, dV = \int_{\mathcal{P}} \rho_0 \frac{\partial^2 \underset{\sim}{f}}{\partial t^2}$$

Now $\int_{\partial \mathcal{P}} \underset{\approx}{S}\, \underset{\sim}{m}\, dS =$ (3 parts)

$$\int_{\Omega} - \underset{\approx}{S}(X_1, X_2, s_1, t)\, \underset{\sim}{e}_3\, dA + \int_{\Omega} \underset{\approx}{S}(X_1, X_2, s_2, t)\, \underset{\sim}{e}_3\, dA$$

$$+ \int_{s_1}^{s_2} \oint_{\partial\Omega} \hat{\underset{\sim}{A}}\, d\ell\, ds$$

imposed traction on lateral surface

Also, $\int_{\mathcal{P}} \rho_0 \frac{\partial^2 \underset{\sim}{f}}{\partial t^2} dV = \int_{s_1}^{s_2} \int_{\Omega} \rho_0 \left(\underset{\sim}{r}_{,tt} + \cancel{X_\alpha\, \underset{\sim}{d}_{\alpha,tt}} \right) dA\, ds$

$\underset{\sim}{0}$ centroidal

Thus (*) reads

$$(**) \quad \left(\int_{\Omega} \underset{\approx}{S}\, \underset{\sim}{e}_3\, dA \right) \Bigg|_{s_1}^{s_2} + \int_{s_1}^{s_2} \left(\int_{\Omega} \hat{\underset{\sim}{b}}\, dA + \oint_{\partial\Omega} \hat{\underset{\sim}{S}}\, d\ell \right) ds$$

$$= \int_{s_1}^{s_2} \left(\int_{\Omega} \rho_0\, dA \right) \underset{\sim}{r}_{,tt}$$

<u>Define</u> $\underset{\sim}{n}(s,t) \equiv \int_{\Omega} \underset{\approx}{S}(X_1, X_2, s, t)\, \underset{\sim}{e}_3\, dA$

"true" contact force

$$\underset{\sim}{\hat{b}}(s,t) \equiv \int_{\Omega} \underset{\sim}{\hat{b}}\, dA + \oint_{\partial\Omega} \underset{\sim}{\hat{s}}\, dl$$

body force/undef. length

$$\rho_0(s) \equiv \int_{\Omega} \rho_0\, dA$$

Then (**) reads:

$$\text{LMB} \quad \boxed{\underset{\sim}{n}\Big|_{s_1}^{s_2} + \int_{s_1}^{s_2} \underset{\sim}{\hat{b}}\, ds = \int_{s_1}^{s_2} \rho_0\, \underset{\sim}{r}_{tt}\, ds}$$

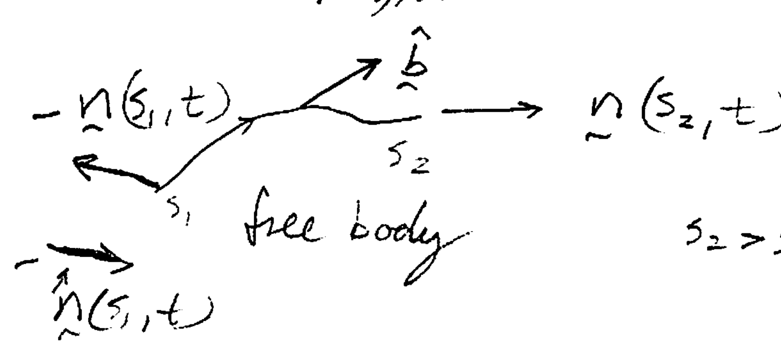
<u>Direct approach</u>

<u>Postulate</u>: contact force $\underset{\sim}{n}(s,t)$, which represents the force at part $\mathcal{P}$ at (s,t) due to contact with "external" world.

could depend on "s"

$\dfrac{\text{body force}}{\text{und. length}}$ $\underset{\sim}{\hat{b}}$, $\dfrac{\text{mass}}{\text{und length}}$ $\rho_0(s)$

Free body



$s_2 > s_1$

$$\text{LMB} \Rightarrow \quad \underset{\sim}{n}(s_2,t) - \underset{\sim}{n}(s_1,t) + \int_{s_1}^{s_2} \hat{\underset{\sim}{b}}\, ds$$
$$= \frac{d}{dt}\int_{s_1}^{s_2} \rho_0\, \underset{\sim}{r}_t\, ds \ \checkmark$$

__Local Form:__

Fund. thm. calc.

Observe $\quad \underset{\sim}{n}\Big|_{s_1}^{s_2} = \int_{s_1}^{s_2} \underset{\sim}{n}_s \, ds$

$$\Rightarrow \int_{s_1}^{s_2} \left(\underset{\sim}{n}_s + \hat{\underset{\sim}{b}} - \rho_0 \underset{\sim}{r}_{tt}\right) ds = \underset{\sim}{0}$$

$\forall \ (s_1, s_2)$ "parts"

Assuming continuity of integrand $\Rightarrow$

$$(*) \quad \boxed{\underset{\sim}{n}_s + \hat{\underset{\sim}{b}} = \rho_0\, \underset{\sim}{r}_{tt}}$$

Next we take the same approach to angular momentum balance: Referring back to our part $\mathcal{P}$ on p. 123, we need the referential version of angular momentum balance:

$$\underset{\sim}{M}_o(\mathcal{P}) = \int_{\partial\mathcal{P}} \underset{\sim}{f} \times \underset{\approx}{S}\, \underset{\sim}{m}\, dS + \int_{\mathcal{P}} \underset{\sim}{f} \times \hat{\underset{\sim}{b}}\, dV$$

total moment

$$= \frac{d}{dt}\, \underset{\sim}{H}_o(\mathcal{P}),$$